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Mathematical Reviews

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FOUNDATIONS

★Łukasiewicz, Jan. Aristotle's syllogistic from the standpoint of modern formal logic. Oxford, at the Clarendon Press, 1951. xii+141 pp. $3.00.

This important book presents a detailed analysis of Aristotle's theory of the syllogism in the light of modern logic. The author first distinguishes the features of the doctrine which are due to Aristotle rather than to later writers. He is careful to note what now appear as inadequacies in Aristotle's treatment. His analysis is both textual and structural, a careful exposition of Aristotle's text being given before the theory is put into a rigorous, deductive formalism. Łukasiewicz's formalization of the theory is especially neat and simple. It contains axioms and rules of rejection as well as the more familiar kind of axioms and rules of assertion. This is in keeping with a rule of rejection stated by Aristotle himself [Prior analytics, i. 5, 27[b] 12–23 (Bekker edition, Reimer, Berlin, 1831)]. Finally, by the addition of a new rule of rejection, it is shown how, following Słupecki [Trav. Soc. Sci. Lett. Wrocław. Ser. B. no. 6 (1948)], every significant expression of Aristotle's syllogistic is either asserted or rejected. This constitutes the solution of a kind of decision problem for this area of logical theory.

R. M. Martin (Philadelphia, Pa.).

Kalmár, László. Contributions to the reduction theory of the decision problem. IV. Reduction to the case of a finite set of individuals. Acta Math. Acad. Sci. Hungar. 2, 125–142 (1951). (Russian summary)

The author shows that the decision problem for the first order predicate calculus is equivalent to the question of whether or not a finite set exists for which a given first order formula is satisfied. In particular, he shows that given a first order formula A, it is possible to construct another first order formula B, such that A is satisfiable if and only if there is no finite set on which B can be satisfied. His proof makes use of linear graphs.

Some consequences are: 1. The class of first order formulas satisfiable on some finite set is non-recursive. 2. The class of first order formulas which are identically true on every finite set is non-recursive (this was proved first by Trachtenbrot). 3. The finitely identical first order formulas are not recursively enumerable. 4. There is no axiom system with a finite number of recursive rules of inference and a recursively enumerable or finite set of axioms, the theorems of which are exactly the finitely identical formulas. 5. The class of first order formulas which are identically true on each finite set but false on some infinite set is not recursively enumerable.

O. Frink (State College, Pa.).

Rice, H. G. Classes of recursively enumerable sets and their decision problems. Trans. Amer. Math. Soc. 74, 358–366 (1953).

Verf. geht von einer Abzählung $\phi_0, \phi_1, \phi_2, \cdots$ aller partiellen rekursiven Funktionen aus. ϵ sei die Menge aller natürlichen Zahlen. Eine Menge $\alpha \subset \epsilon$ wird durch ϕ_r aufgezählt, wenn für alle a gilt: $a \in \alpha \leftrightarrow$ für ein b: $\phi_r(b)=a$. Die Menge α heisst rekursiv aufzählbar (r.a.), wenn es ein r gibt, so dass α durch ϕ_r aufgezählt wird. α heisst rekursiv (r.), wenn α und $\epsilon-\alpha$ r.a. sind. Für jede Klasse A von r.a. Mengen sei θ_A die Menge der r, für die es ein $\alpha \in A$ gibt, so dass α durch ϕ_r aufgezählt wird. A heisst vollständig rekursiv aufzählbar (v.r.a.) bzw. vollständig rekursiv (v.r.) wenn θ_A r.a. bzw. r. ist.

Für jede Zahl k ist die Klasse $L(k)$ der r.a. Mengen α mit $k \in \alpha$ v.r.a. Verf. vermutet, dass jede v.r.a. Klasse als $\bigcup_{m\epsilon\mu}\bigcap_{n\epsilon\nu}L(k_{mn})$ mit r.a. μ und endlichem ν darstellbar ist. Es wird bewiesen, dass (1) jede nicht-leere v.r.a. Klasse mindestens eine endliche Menge enthält und (2) jede v.r.a. Klasse mit einer endlichen Menge α auch alle Obermengen von α enthält. Hieraus folgt leicht, dass eine v.r. Klasse entweder leer ist oder alle r.a. Mengen enthält. Dieses Resultat bleibt gültig, wenn zur Aufzählung der r.a. Mengen nur allgemein-rekursive oder primitiv-rekursive Funktionen zugelassen werden.

P. Lorenzen.

Péter, Rózsa. Probleme der Hilbertschen Theorie der höheren Stufen von rekursiven Funktionen. Acta Math. Acad. Sci. Hungar. 2, 247–274 (1951). (Russian summary)

Recursive functions of Order I are defined using only natural number variables, those of Order II may require, in addition, variables for functions of type of Order I. The author proves that every function of Order I defined by a k-fold nested recursion may be defined by k primitive recursions of Order II. Conversely, every primitive recursive function of Order II may be expressed as a recursive function of Order I. This result may be extended to nested recursions of Order II in which the nestings occur only through substitutions on natural number variables. It is not known whether there exist functions of Order II which are not multiple recursive of Order I. However, the author shows that any multiple recursive function of Order II may be expressed in terms of a function of natural number variables of a variable number of arguments; such a function may also be defined by transfinite recursion of ordinal type ω^ω. The diagonal function ψ of all multiple recursive functions of Order I is also a function of this type, but the author observes that ψ is completely multiple (vollzählig-mehrfach) while the former are scatteredly multiple (zerstreut-mehrfach). A function is scatteredly multiple in case there exists a k such that the computation of the value of the function of r arguments depends upon the computation of the values of the function for lesser arguments in at most k places, possibly different places for different arguments, but k constant with respect to r. If the recursion runs along all r of the variables, the function is said to be completely multiple. The author also notes that the Ackermann majorization argument offers no help for the unsolved problem.

D. Nelson (Washington, D. C.).

Beth, E. W. Some consequences of the theorem of Löwenheim-Skolem-Gödel-Malcev. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 66–71 (1953).

In a previous paper [same Proc. **54**, 436–444 (1951); these Rev. **13**, 614], the author gave a topological proof of the well-known Skolem-Loewenheim theorem in the following general form first proved by Malcev [Mat. Sbornik N.S. **1**(43), 323–336 (1936)]: A set $\mathfrak{a}$ of closed expressions containing Z predicate and individual parameters has a model of cardinal number Z at most if and only if the system $K(\mathfrak{a})$ of all expressions derivable from $\mathfrak{a}$ by means of the first order predicate calculus with identity is consistent (if Z is finite, then the model may be denumerable). The proof of this uses the axiom of choice. The present paper deals with some consequences of this theorem.

Let $\mathfrak{a}$ be any complete consistent system of first order functional calculus with identity, M a model of $\mathfrak{a}$. Let $\mathfrak{b}$ be any set of matrices with x as their only free variable and $M(\mathfrak{b})$ the set of all elements of M which satisfy all matrices $A(x)$ in $\mathfrak{b}$. Then: 1) if $\mathfrak{b}$ is a finite set, $M(\mathfrak{b})$ will be empty if and only if $\mathfrak{b}$ is inconsistent with $\mathfrak{a}$; 2) if $\mathfrak{b}$ is infinite and consistent with $\mathfrak{a}$, then we can always find a model M' of $\mathfrak{a}$ such that $M'(\mathfrak{b})$ is non-empty; 3) $\mathfrak{a}$ has a model M_0 such that $M_0(\mathfrak{b})$ is empty if and only if $\mathfrak{b}$ is inconsistent with $\mathfrak{a}$, that is, if and only if, for every model M of $\mathfrak{a}$, the corresponding set $M(\mathfrak{b})$ is empty. The next theorem states: Suppose we have a set M and a binary relation R and that for some natural number p every finite subset M' of M can be divided into p disjoint sets $K_1', \cdots, K_p'$ such that any two elements u and v of K_m' $(m \leq p)$ are in the relation R; then M itself admits a similar division into p disjoint sets $K_1, \cdots, K_p$.

Let $\mathfrak{a}$ be a set of closed sentences of first order predicate calculus containing only one binary predicate parameter r and only prenex universal quantifiers. Suppose M is a set on which a binary relation R is defined and that $C \subseteq M$. If R restricted to C satisfies all statements of $\mathfrak{a}$, then C is called an $(R, \mathfrak{a})$ chain. We then have: Every $(R, \mathfrak{a})$ chain C in a set M is contained in a maximal $(R, \mathfrak{a})$ chain C' in M. This theorem and generalizations of it give as corollaries several equivalents of the axiom of choice.

I. Novak Gál (Ithaca, N. Y.).

Dienes, Z. P. Sulla definizione dei gradi di rigore. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 223–253 (1952).

The author accepts the intuitionistical propositional calculus and the predicate calculus in which every atomic predicate is decidable. He distinguishes four degrees of rigour, which we denote by (i) to (iv). If $F(x)$ is a decidable predicate, then $(x)F(x)$ is considered as decidable in (i) and (ii) if x ranges over a finite set, in (iii) and (iv) if x ranges over a finite or denumerable set; $(Ex)F(x)$ is considered as decidable in (i) if x ranges over a finite set, in (ii) and (iii) if x ranges over a finite or denumerable set, and in (iv) if x ranges over a set of cardinal number at most $\aleph_1$. Several examples are given. Equality of real numbers belongs to (iii), while inequality can be treated in (ii). The construction of the limit of a bounded monotone sequence of real numbers can be performed in (iii). The problem of whether a given inner limiting set is void or not is solved in (iv). The author identifies his first three degrees with intuitionism, theory of general recursive functions, and a system of Goodstein [Trans. Amer. Math. Soc. **68**, 174–182 (1950); these Rev. **12**, 233]; in reality, each of these conceptions overlaps parts of different degrees.

A. Heyting (Amsterdam).

Scholz, Heinrich. Der klassische und der moderne Begriff einer mathematischen Theorie. Math.-Phys. Semesterber. **3**, 30–47 (1953).

Expository lecture.

Routledge, N. A. Ordinal recursion. Proc. Cambridge Philos. Soc. **49**, 175–182 (1953).

$\prec$ sei eine Wohlordnung der natürlichen Zahlen mit (1) n_0 als erstem Element und (2) $g(x) \prec x$ für $x \neq n_0$ und eine primitiv-rekursive (p.r.) Funktion g. In Verallgemeinerung des p.r. Definitionsschemas definiert dann

$$(3) \qquad \begin{aligned} f(n_0, \mathfrak{y}) &= h(\mathfrak{y}), \\ f(x, \mathfrak{y}) &= \phi(f(g(x), \mathfrak{y}), x, \mathfrak{y}) \quad \text{für } x \neq n_0 \end{aligned}$$

eine Funktion f. Sind auch h und ϕ p.r., dann heisst f ordinal-rekursiv (o.r.). Verf. beweist, dass jede allgemein-rekursive (a.r.) Funktion F darstellbar ist durch $F(\mathfrak{x}) = f(\phi(\mathfrak{x}))$ mit p.r. ϕ und o.r. f. Dagegen ist nicht jede a.r. Funktion o.r.

Als Typen der Wohlordnung $\prec$ braucht nur ω zugelassen zu werden, denn (3) definiert genau dann eine Funktion, wenn für die iterierten Funktionen $g^2(x) = g(g(x))$, $g^3(x) = g(g^2(x))$, $\cdots$ gilt (4) $\bigwedge_x \bigvee_m g^m(x) = n_0$. Zu jeder Funktion g, die (4) erfüllt, gibt es aber eine Wohlordnung vom Typ ω, die (1) und (2) erfüllt.

P. Lorenzen (Bonn).

Takeuti, Gaisi. A metamathematical theorem on the theory of ordinal numbers. J. Math. Soc. Japan **4**, 146–165 (1952).

It is shown in this paper that the consistency proof of a theory of ordinal numbers in the weakened form considered in Gentzen's system LK can be reduced to that of a weakened theory of ordinal numbers $< \omega^\omega$, this latter theory being considered in a logical system which is obtained by extending the system LK slightly by the use of the symbol Min: if $\mathfrak{A}(a)$ is a formula and x is any bound variable not contained in $\mathfrak{A}(a)$, the figure Min $(x)\mathfrak{A}(x)$ is a term.

I. Novak Gál (Ithaca, N. Y.).

Rohrbach, Hans. Das Axiomensystem von Erhard Schmidt für die Menge der natürlichen Zahlen. Math. Nachr. **4**, 315–321 (1951).

This paper gives an account of Erhard Schmidt's characterization of the natural numbers which was used in his lectures from 1920 on, but never published. Later, quite independently, Kaczmarz published essentially the same axioms [J. London Math. Soc. **7**, 179–182 (1932)]. Schmidt defines the set of natural numbers as a non-empty ordered set $\mathfrak{N}$ such that: (1) every non-empty subset of $\mathfrak{N}$ has a first element; (2) every element of $\mathfrak{N}$ except the first has an immediate predecessor in $\mathfrak{N}$; (3) $\mathfrak{N}$ has no last element. It is shown that this is equivalent to the Dedekind-Peano axioms. Kaczmarz gave these axioms in a form involving the $<$ relation alone. It is unfortunate that these simple axioms for the natural numbers are not better known.

I. Novak Gál (Ithaca, N. Y.).

Lorenzen, Paul. Über den Mengenbegriff in der Topologie. Arch. Math. **3**, 377–386 (1952).

Verf. wirft die Frage auf, inwieweit in der sogenannten mengentheoretischen Topologie der Mengenbegriff wesentlich verwendet wird. Ein Gebiet heisse elementar formalisierbar, wenn keine Prädikatenvariabelen quantifiziert werden. Man kann weitgehend mit elementaren Methoden auskommen (z.B. Einbettung in den Hilbertschen Raum ausführen), wenn man Mengen als neue Individuen neben den Punkten und die Adhärenz als undefinierte Relation einführt.

H. Freudenthal (Utrecht).

Gokieli, L. P. On paradoxes of the theory of sets. Soobščeniya Akad. Nauk Gruzin. SSR 9, 3–10 (1948). (Russian)

The author contends that the paradoxes are easily solved by the philosophy of K. Marx, which affirms the unity of a general notion with its particular instances. The discussion is entirely philosophical; formalism, and, as it seems, also formalization, is rejected as the expression of a "bourgeois" philosophy which hypostases the notion of a set.

A. Heyting (Amsterdam).

Dieudonné, Jean. Logic and mathematics. Revista Mat. Elem. 2, 1–7 (1953). (Spanish)

Lecture given at the Universitad de los Andes in October 1952.

Apéry, R. Les mathématiques sont-elles une théorie pure? Dialectica 6, 309–310 (1952).

Bouligand, G. La pensée prospective en mathématiques. Dialectica 6, 305–308 (1952).

Geymonat, Ludovico. Significato filosofico-scientifico delle ricerche moderne sugli spazi astratti. Archimede 5, 1–8 (1953).

***Hasse, Helmut. Mathematik als Wissenschaft, Kunst und Macht.** Verlag für Angewandte Wissenschaften, Wiesbaden, 1952. 34 pp.

Lecture given at the University of Hamburg in January, 1951. The author treats subjectively the nature of mathematics, aesthetic aspects of mathematics and of mathematical exposition, and various other aspects of mathematical experience.

ALGEBRA

Kishen, K., and Rao, C. R. An examination of various inequality relations among parameters of the balanced incomplete block design. J. Indian Soc. Agric. Statistics 4, 137–144 (1952).

The inequality $b \geqq v$ for the parameters of an incomplete balanced block design has first been proved by R. A. Fisher. R. C. Bose [Sankhyā 6, 105–110 (1942); these Rev. 4, 237] and K. R. Nair [ibid. 6, 255–259 (1943); these Rev. 5, 29] improved this inequality for resolvable designs with r replications to $b \geqq v+r-1$, $b \geqq rK(r-1)/[(r-K+\lambda(K-1)]$, respectively. The author gives two new inequalities but shows that his inequalities as well as Bose's and Nair's are implied by Fisher's result and by the relations $r(K-1)=\lambda(v-1)$, $v=nK$, $b=nr$ (n integral), which must trivially hold for a resolvable design, n being the number of blocks which constitute a complete replication.

H. B. Mann.

Green, Bert F. The orthogonal approximation of an oblique structure in factor analysis. Psychometrika 17, 429–440 (1952).

Given two $k \times m$ matrices $(k \geqq m)$, A and B, such that $A'B$ is of rank m, the problem is to find an orthogonal transformation Λ $(\Lambda'\Lambda = I)$ which minimizes the sum of squares of elements in the difference matrix $(A\Lambda - B)$. The solution is $\Lambda = PD^{-1/2}P'$, where the columns of P are the latent vectors (principal axes) of the matrix $(A'BB'A)$ and the diagonal matrix D has as elements (i, i) the latent roots of $(A'BB'A)$. Arbitrary weights may be introduced in this general solution. The factor analysis problem is to apply Λ to the oblique (factor) matrix A and come "as close as possible" to B. Three analytical methods of determining Λ are considered.

R. L. Anderson (Raleigh, N. C.).

Taussky, Olga, and Todd, John. Systems of equations, matrices and determinants. Math. Mag. 26, 9–20, 71–88 (1952).

The first chapter of this expository paper is an exceedingly compact survey of the basic theorems of matrix algebra. There are also allusions to results not usually included in an introductory text. Thus the incidence matrices of a projective plane are mentioned as instances of normal matrices and there is a brief discussion of bounds for the characteristic values. Although it is entirely consonant with the purpose of the paper, the use of only the real and complex fields and the brevity of statement could mislead an unwary student in a few places: "A real unitary matrix is called orthogonal" (p. 12), which must not be construed as a definition since the orthogonal group is defined over any field; "Skew symmetric matrices are real matrices (a_{ik}) with $a_{ik} = -a_{ki}$" (p. 17), which is subject to the same objection. In the matrix at the bottom of page 16, the subscripts should be elements. There is no use of the concept of a vector space, so that matrices are treated formally rather than as describing mappings.

This is in accordance with the purpose of the second chapter which treats, with specific numerical examples, computational methods for solving systems of linear equations and for finding characteristic values. The elimination method is explained, the determinantal solution is discarded and a finite iteration scheme, amounting to finding the center of an ellipse, is illustrated. Indirect solutions include relaxation, "Seidel" iteration and gradient methods. Cholesky's and an iteration scheme are used to calculate an inverse matrix. Straightforward iteration and relaxation using Rayleigh's approximation give a dominant characteristic root. Jacobi's method of reducing the matrix of a real quadratic form to a diagonal matrix by a sequence of rotations in coordinate planes is described. Here a student might have been warned that the practical use of the method requires a (small integral?) multiple of $\frac{1}{2}n(n-1)$ plane rotations and that three rotations will usually not suffice for a matrix of order four as in the example. The paper closes with a valuable note of warning by showing how rounding-off errors may seriously effect accuracy when the coefficient matrix is ill-conditioned. The paper can be especially recommended to those interested in matrix algebra for its use in computing.

W. Givens (Knoxville, Tenn.).

***Ferrar, W. L. Finite matrices.** Oxford, at the Clarendon Press, 1951. vii+182 pp. $4.00.

This elementary text is considerably more ambitious than some. It discusses the equivalence problems $P\,A\,P^{-1}$, $P\,A\,P^{*}$ in an adequate manner, the base field being complex numbers, or in places a more general ring. There is a chapter on infinite series and functions of matrices which concerns i.a. finding (some) fractional powers of matrices. The last chapter develops (a) a series for $(\lambda I - A)$ for large λ; (b) matrices E_i such that $E_iE_j + E_jE_i = 0$ $(i \neq j)$, $E_i^2 = -I$, of interest in quantum mechanics; (c) the condition that a real quadratic form $\sum a_{ij}x_ix_j$ be positive definite if the x_i

satisfy $\sum p_{ij}x_i=0$ $(j=1, 2, \cdots, t)$. The general theory of linear transformations over vector spaces is exploited little, except in solving $X^n=A$. No exercises appear. There is a good index. The transpose of (a_{rs}) is incorrectly written as (a_{sr}). *J. L. Brenner* (Pullman, Wash.).

Tchernikow, S. N. A generalization of the Kronecker-Capelli theorem on a system of linear equations. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif., NBS Rep. **2346.** i+19 pp. (1953).

Translated from Mat. Sbornik N.S. **15**(57), 437–448 (1944); these Rev. **7**, 109.

Wiegmann, N. A. A note on pairs of normal matrices with property L. Proc. Amer. Math. Soc. **4**, 35–36 (1953).

If A and B are square matrices such that $\alpha A+\beta B$ has roots $\alpha\lambda_i+\beta\mu_i$ for a suitable ordering of the roots λ_i of A and μ_i of B, then if they are normal, they must commute. *W. Givens* (Knoxville, Tenn.).

Carlitz, L. A note on orthogonal matrices. Amer. Math. Monthly **60**, 253–255 (1953).

Duparc, H. J. A., and Peremans, W. An observation on rapport ZW 1949-001. Math. Centrum Amsterdam. Rapport **ZW 1952-020**, 3 pp. (1952). (Dutch)

The inverse of the matrix (c_{rs}) of order $t+1$, where $c_{rs}=\binom{s+r-1}{t+s}$ has been computed by Korevaar and Scheelbeek [Math. Centrum Amsterdam. Rapport **ZTW 1949-001** (1949); these Rev. **11**, 3]. In this paper the inverse of this matrix is obtained rapidly as a special case of a general formula proved earlier by van der Corput and Duparc [Indagationes Math. **8**, 615–622, 671–674 (1946); these Rev. **8**, 308]. *A. W. Goodman* (Lexington, Ky.).

on order

***Bergström, Harald. A triangle-inequality for matrices.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 264–267. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let A and B be two symmetric matrices of real elements. Denote by A_{11} and B_{11}, respectively, the first principal minors. Assume they are both positive definite. The author proves that

$$\frac{|A+B|}{|A_{11}+B_{11}|}\geq\frac{|A|}{|A_{11}|}+\frac{|B|}{|B_{11}|}$$

where $|A|$ denotes the determinant of A. The author discusses applications in probability theory. *P. Erdős.*

Fomin, S. V. The basic concepts of linear algebra. Mat. v Škole **1953**, no. 1, 1–15 (1953). (Russian)

Expository paper.

Aitken, A. C. A note on trace-differentiation and the Ω-operator. Proc. Edinburgh Math. Soc. (2) **10**, 1–4 (1953).

If $X=[x_{ij}]$ is an $n\times n$ matrix of n^2 independent elements x_{ij}, and if $\Omega=[\partial/\partial x_{ji}]$ is a matrix of differential operators whose ijth element is $\partial/\partial x_{ji}$, then Ω has important properties in classical invariant theory, dating back to Cayley's determinantal theorm:

$$(1)\qquad |\Omega|\,|X|^r=r(r+1)(r+2)\cdots(r+n-1)|X|^{r-1}.$$

The present note shows how to extend the properties to the symmetric case $X=X'$ by taking a suitable modified form of the matrix of operators:

$$\Omega=[\epsilon_{ij}\partial/\partial x_{ji}],\quad \epsilon_{ii}=1,\ \epsilon_{ij}=p_{ji}=\tfrac{1}{2},\ i\neq j.$$

Such extensions forced themselves on the notice of the author [Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 369–377 (1948), p. 374; these Rev. **10**, 201] in some work on the estimation of statistical parameters. Simultaneously and independently Gårding obtained the analogue of (1) in the form

$$(3)\qquad |\Omega|\,|X|^r=r(r+\tfrac{1}{2})(r+1)\cdots(r+\tfrac{1}{2}n-\tfrac{1}{2})|X|^{r-1}$$

where $X=X'$ and Ω is given by (2) [Proc. Edinburgh Math. Soc. (2) **8**, 73–75 (1948); these Rev. **10**, 347].

Let Ωf denote the matrix whose ijth element is $\epsilon_{ij}\partial f/\partial x_{ji}$ (with $\epsilon_{ij}=1$ in the original case); in particular, if f is the trace $S(Y)$ of a matrix Y, let the expression be written $\Omega_S Y$. By trace-differentiation is meant this operator Ω_S, whose fundamental property is

$$(4)\qquad \Omega_S X^r=rX^{r-1},\quad \Omega_S X^{-r}=-rX^{-r-1}$$

[Turnbull, ibid. (2) **2**, 33–54 (1930); 256–264 (1931)], the latter of which the author proves anew (for positive integral r) by an ingenious appeal to cyclic symmetry. He then extends and establishes (4) for the symmetric case $(x_{ij}=x_{ji})$, and lastly for the skew-symmetric case (with even values of r) by taking $\epsilon_{ii}=0$, $\epsilon_{ij}=\epsilon_{ji}=\tfrac{1}{2}$, $i\neq j$. *H. W. Turnbull.*

Foulkes, H. O. Matrix differentiation of *S*-functions. Proc. Edinburgh Math. Soc. (2) **10**, 5–10 (1953).

Let $X=[x_{st}]$ be an n-rowed square matrix, S_r, a_r, h_r symmetric functions of the latent roots, $\{\lambda\}$ an S-function of the latent roots, and $\Omega=[\partial/\partial x_{ts}]$ a matrix differential operator. The following results are known [H. W. Turnbull, Proc. Edinburgh Math. Soc. (2) **1**, 111–128 (1928)]:

$$\Omega S_r=rX^{r-1},$$
$$\Omega a_r=(-1)^{r-1}\{X^{r-1}-a_1X^{r-2}+\cdots+(-1)^{r-1}a_{r-1}\},$$
$$\Omega h_r=X^{r-1}+h_1X^{r-2}+\cdots+h_{r-1}.$$

The author shows that

$$\Omega\{\lambda\}=\frac{\partial\{\lambda\}}{\partial S_1}+2\frac{\partial\{\lambda\}}{\partial S_2}X+\cdots+n\frac{\partial\{\lambda\}}{\partial S_n}X^{n-1}.$$

He shows how to evaluate this by expressing the operator $\partial/\partial S_r$ in terms of operators $D_{(\mu)}$ associated with S-functions.

The work would have been simplified, however, if he had used the result

$$\frac{\partial\{\lambda_1, \lambda_2, \cdots, \lambda_r\}}{\partial S_i}=\{\lambda_1-i, \lambda_2, \cdots, \lambda_r\}$$
$$+\{\lambda_1, \lambda_2-i, \cdots, \lambda_r\}+\cdots+\{\lambda_1, \lambda_2, \cdots, \lambda_r-i\},$$

the S-functions on the right being interpreted according to Murnaghan's convention [see, e.g., Proc. Nat. Acad. Sci. U. S. A. **38**, 738–761 (1952); these Rev. **14**, 244] when the parts are not in descending order.

Some further results of Turnbull [loc. cit.], namely,

$$\Omega^2h_p=(n+p-1)\Omega h_{p-1},\quad \Omega^2a_p=-(n-p+1)\Omega a_{p-1},$$

are generalised. The author shows that

$$\Omega^r\{p\ 1\}=(n+p-1)\Omega^{r-1}\{p-1, 1\},\quad p>0,\ r\geq 2.$$

Also, putting, e.g.,

$$(p, 4)=\{p\ 4\}+\{p\ 3\ 1\}+\{p\ 2^2\}+\{p\ 2\ 1^2\}+\{p\ 1^4\},\quad p>4,$$

then

$$\Omega^r(p, m)=(n+p-1)\Omega^{r-1}(p-1, m),\quad r\geq m+2.$$

There are similar results for conjugate partitions. *D. E. Littlewood* (Bangor).

Duparc, H. J. A. On canonical forms. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 474–482 (1952).

The author generalizes the Lasker-Wakeford Theorem on canonical forms [E. Lasker, Math. Ann. **58**, 434–440 (1904); E. K. Wakeford, Proc. London Math. Soc. (2) **18**, 403–410 (1920)], and thereby extends its scope in applications to simultaneous systems of forms.

Let $f_1(x), \cdots, f_k(x)$ be k forms, each of the type $f(x)=\sum a_{ij\ldots}x_ix_j\cdots=(a'x)^n$, homogeneous and of degree n in m cogredient variables x_i, and with N coefficients $a_{ij\ldots}$. Let $\phi(u)=(\alpha u')^n$ be a form of the same type but in m contragredient variables u_i. The f_i are called similar forms, and $\phi(u)$ a dual similar form. The f and ϕ are apolar if their lineo-linear invariant $(a'\alpha)^n$ vanishes: forms f and $\psi=(\alpha u')^{n'}$, of differing degrees n, n' are apolar if $(a'\alpha)^{n'}(a'x)^{n-n'}=0$ identically in x when $n>n'$ and if $(a'\alpha)^n(\alpha u)^{n'-n}=0$ identically in x when $n<n'$.

Let $f(x)=F(X, c)$ denote the form f expressed as a form in r variables $X_1, \cdots, X_r$ and s parameters $c_1, \cdots, c_s$ (all depending on the original N coefficients and the variables x_i) such that (1) $X_\rho=X_\rho(x, e)=\sum e_{\rho, ij\ldots}x_ix_j\cdots$ $(\rho=1, \cdots, r)$ where the total number t of the distinct coefficients $e_{\rho, ij\ldots}$ satisfies the relation $t+s\geqq N$. Further, let $\phi(u)$, which is dual similar to $f(x)$, denote a form which is apolar to each of the $r+s$ forms $\partial F/\partial X_\rho$, $\partial F/\partial c_\sigma$ $(\rho=1, \cdots, r; \sigma=1, \cdots, s)$. The Lasker-Wakeford Theorem asserts that $F(X, c)$ is a legitimate canonical representation of $f(x)$ if such an apolar form $\phi(u)$ fails to exist for some choice of the $t+s$ parameters $c_1, \cdots, c_s, e_1, \cdots, e_t$. The author states and proves the following extension: If k similar forms $f_1(x), \cdots, f_k(x)$, each possessing N coefficients, have the simultaneous representation $F_1(X, c), \cdots, F_k(X, c)$ involving altogether s parameters $c_1, \cdots, c_s$ and r variables $X_1, \cdots, X_r$ satisfying (1), the total number t of coefficients satisfying $t+s\geqq kN$, then there exists a set of numbers $\lambda_1, \cdots, \lambda_k$ (not all zero) such that a dual similar form $\phi(u)$, which is apolar to all $r+s$ forms $\partial F_\lambda/\partial X_\rho$, $\partial F_\lambda/\partial c_\sigma$ $(\rho=1, \cdots, r; \sigma=1, \cdots, s)$ with $F_\lambda=\sum_{\kappa=1}^k\lambda_\kappa F_\kappa(X, c)$, does not always exist. If, however, the set of forms $F_1, \cdots, F_k$ is not a legitimate simultaneous representation of the $f_1, \cdots, f_k$, then there exists such a set λ_i for which $\phi(u)$ always exists.

Various applications are given: e.g., two general binary cubics f and g can be brought simultaneously to the forms

$$F=X_1^3+X_2^3+X_3^3, \quad G=c_1X_1^3+c_2X_2^3$$

in terms of three linear forms X_i and two distinct, nonzero c_i. Two binary n-ics can be brought simultaneously to the form

$$(2) \qquad F=\sum_{i=1}^D X_i^n, \quad G=\sum_{i=1}^D c_iX_i^n \quad (c_i\neq 0, c_i\neq c_j, i\neq j)$$

where D denotes the smallest integer $\geqq[n/2]+2$ and $\geqq\frac{2}{3}(n+1)$. A binary sextic can be brought to the form $(X_1^3+X_2^3+X_3^3)(c_1X_1^3+c_2X_2^3)$. Two general m-ary n-ics can be brought to the form (2) where each X_i is a linear form in m variables and where D is the smallest integer $\geqq d_m(n)+2$ and $\geqq(2/(m+1))\binom{n+m-1}{n}$, where $d_m(n)$ is the greatest number of points through which an m-ary manifold of degree n can be drawn, having a double point at each of these points.

H. W. Turnbull (Cambridge, England).

Plans, Antonio. On the metric affine invariants of quadratic forms. Gaceta Mat. (1) **4**, 248–253 (1952). (Spanish)

Iseki, K. On the conjugate mapping for quaternions. Publ. Math. Debrecen **2**, 204–205 (1952).

Let $x\rightarrow x'$ be a mapping over real quaternions. Theorem 2. x' is the conjugate, (norm x)$/x$, of x if (i) norm x' is continuous at $x=0$; (ii) $xx'=x'x$; (iii) xx', $x+x'$ are real.

J. L. Brenner (Pullman, Wash.).

Abstract Algebra

Inaba, Eizi. Some remarks on primary lattices. Nat. Sci. Rep. Ochanomizu Univ. **2**, 1–5 (1951).

Following the author's previous paper [J. Fac. Sci. Hokkaido Univ. Ser. I. **11**, 39–107 (1948); these Rev. **10**, 348], the author gives further characterizations of "primary", finite-dimensional modular lattices. One such characterization: every interval is indecomposable. Another: for any $b>a$, b is a join of join-irreducible elements which are independent over a. *G. Birkhoff* (Cambridge, Mass.).

Iseki, K. Contribution to lattice theory. Publ. Math. Debrecen **2**, 194–203 (1952).

The author collects various results, a few of them apparently new, on characterizations of distributive lattices and Boolean algebras, on representations of lattices by sets, and on Brouwerian algebras. Sample theorems: A lattice is distributive if and only if $a=b$ is implied by $a\leqq c\cup d$, $b\leqq c\cup d$, $a\cap c=b\cap c$, and $(a\cup c)\cap d=(b\cup c)\cap d$. An atomic lattice in which if $a>b$ then, for some x, $a\cap x\neq 0$ and $b\cap x=0$ is isomorphic with the lattice of all sets of its atoms [cf. Theorem 14 of G. Birkhoff and O. Frink, Trans. Amer. Math. Soc. **64**, 299–316 (1948); these Rev. **10**, 279]. In a Boolean algebra, the last residue class of the last residue class of an ideal J is J, and dually.

The bibliography includes several little known works, but the author's indications of priority are not always correct. E.g., for sum- and product-complements, see H. M. MacNeille, Trans. Amer. Math. Soc. **42**, 416–460 (1937); for cases of p-separation (though not explicitly named), see G. Birkhoff, Proc. Cambridge Philos. Soc. **29**, 441–464 (1933) and Duke Math. J. **3**, 443–454 (1937), M. H. Stone, Časopis Pěst. Mat. Fys. **67**, 1–25 (1937), and the reviewer, Amer. J. Math. **65**, 179–196 (1943); these Rev. **4**, 129.

P. M. Whitman (Silver Spring, Md.).

Iséki, Kiyoshi. A criterion for distributive lattices. Acta Math. Acad. Sci. Hungar. **3**, 241–242 (1952). (Russian summary)

A simpler proof that a lattice is distributive if and only if every meet-irreducible ideal is prime [cf. Iseki, C. R. Acad. Sci. Paris **230**, 1726–1727 (1950); these Rev. **12**, 75].

P. M. Whitman (Silver Spring, Md.).

Dilworth, R. P., and McLaughlin, J. E. Distributivity in lattices. Duke Math. J. **19**, 683–693 (1952).

A generalized kind of distributivity in lattices called ϕ-distributivity is defined and studied. This means distributivity with respect to an imbedding operator ϕ. The notion of an imbedding operator is due to M. Ward, and means a closure operator on the subsets of a lattice which sends single-element sets into their principal ideals. Particular cases of ϕ-closed sets are ideals, σ-ideals, complete ideals, and normal ideals. The corresponding distributivity conditions are studied here, with emphasis on normal distributivity, which is the strongest type.

Some of the principal results are: 1. A complete lattice is normally distributive if and only if it is infinitely distributive. 2. The normal completion of a lattice L is infinitely distributive if and only if L is normally distributive. 3. The normal completion of the lattice of bounded continuous functions on a topological space is infinitely distributive. 4. All Boolean algebras are normally distributive. 5. Not every infinitely distributive lattice is normally distributive.
O. Frink (State College, Pa.).

★**Aubert, K. E. Lattice-theoretic aspects of abstract ideal theory.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 243–254. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Mainly expository. The author suggests that, for some applications, a residuated lattice with subtraction may be appropriate. *G. Birkhoff* (Cambridge, Mass.).

Isiwata, Takesi. Linearization of topological groups and ordered rings. Kōdai Math. Sem. Rep. **1952**, 33–35 (1952).

The rings of integers and of real numbers are characterized abstractly within the class of simply ordered rings. Thus, for the integers, the presence of a strong unit and of a positive atomic element is sufficient. *G. Birkhoff.*

Vorob'ev, N. N. Associative systems of which every subsystem has a unity. Doklady Akad. Nauk SSSR (N.S.) **88**, 393–396 (1953). (Russian)

The author considers associative systems (i.e., semigroups) $\mathfrak{S}$ such that every subsystem (i.e., subsemigroup) of $\mathfrak{S}$ contains a unity (two-sided identity element). He shows that a system $\mathfrak{S}$ has this property if and only if (1) $\mathfrak{S}$ is the class sum of mutually disjoint groups in each of which every element has finite order, and (2) the set of idempotent elements of $\mathfrak{S}$ is well-ordered by the division relation (E_1 divides E_2 if $E_1E_2=E_2E_1=E_2$). From this and a theorem of the reviewer [Ann. of Math. (2) **42**, 1037–1049 (1941); these Rev. **3**, 199] it follows that the structure of such a system $\mathfrak{S}$ can be described explicity [the semi-lattice P in the reviewer's Theorem 3 being in this case a well-ordered set]. This enables the author to describe the ideals and automorphisms of $\mathfrak{S}$. A system $\mathfrak{S}$ in which every finitely generated subsystem of $\mathfrak{S}$ contains a unity is similarly characterized, the only change being to substitute "(totally) ordered" for "well-ordered" in (2). *A. H. Clifford.*

Pickert, Günter. Direkte Zerlegungen von algebraischen Strukturen mit Relationen. Math. Z. **57**, 395–404 (1953).

The author examines algebraic systems with a family P of algebraic operations which may involve the combination of an arbitrary number of elements; however, one of the operations shall be an addition $x+y$ with a zero element. The author introduces as permissible operators on the system those which commute with the operations in P. By means of permissible projection endormorphisms, direct decompositions are defined for the system and, based on a theorem by the reviewer on direct decompositions in lattices, an analogue of the Remak direct decomposition theorem for groups is obtained. *O. Ore* (New Haven, Conn.).

Osima, Masaru. On the basic ring. Sûgaku **4**, 138–145 (1952). (Japanese)

Let A be a ring with unit element and assume that the (finite) Remak-Schmidt theorem holds for its decomposition into left ideals. Let e be the sum of a maximal system of mutually orthogonal, mutually non-isomorphic primitive idempotent elements in A; two idempotent elements are called isomorphic when they generate isomorphic left-ideals. The subring eAe of A is then called a basic ring of A. To this notion is given a systematic treatment, which amplifies some results by Brauer (unpublished), Nesbitt and Scott [Ann. of Math. (2) **44**, 534–553 (1943); these Rev. **5**, 89], the reviewer [ibid. **40**, 611–633 (1939); these Rev. **1**, 3] and others, and which includes a study of basic rings of the endomorphism ring of a module. *T. Nakayama.*

Wolfson, Kenneth G. An ideal-theoretic characterization of the ring of all linear transformations. Amer. J. Math. **75**, 358–386 (1953).

$T(F, A)$ denotes the ring of all linear transformations of a vector space A over a not necessarily commutative field F; $T_\nu(F, A)$ is the subring of all linear transformations of A of rank $<\aleph_\nu$. In particular, $T_0(F, A)$ is the set of all linear transformations of A of finite rank. If K is any ring, a left (right) annulet is the left (right) ideal consisting of all left (right) annihilators of a subset of K. The socle K_0 of K is the intersection of all the minimal right ideals of K, if any exist; otherwise, K_0 is the zero ideal.

First it is shown that an abstract ring K is isomorphic to $T_0(F, A)$ for some F and A if and only if: (1) K is a simple ring (not a zero-ring) containing minimal right ideals; (2) if H is a left ideal of K without non-zero right annihilators, then $H=K$. Next, an ideal-theoretic characterization of $T_\nu(F, A)$ is given. The main theorem of the paper states that a ring K is isomorphic to $T(F, A)$ for some F and A if and only if: (1) the socle K_0 of K is not a zero-ring, and is contained in every non-zero two-sided ideal of K; (2) if J is a left ideal of K annihilated on the right only by zero, then $J\supset K_0$; (3) the sum of two left (right) annulets is a left (right) annulet; (4) K has an identity element.

Two rings K and K' are said to have the same ideal theory if there is a mapping ϕ which is simultaneously a projectivity of the set of left, right, two-sided ideals of K upon respectively the set of left, right, two-sided ideals of K', and which also satisfies $(LR)^\phi=L^\phi R^\phi$, for any left ideal L and right ideal R of K. The author shows that if A has rank at least three over F, then a ring K has the same ideal theory as $T(F, A)$ if and only if K and $T(F, A)$ are isomorphic. Examples are given which show the condition that A have rank at least three over F is necessary.

Many of the lemmas and theorems used to obtain the characterizations cited above are of interest for their own sake, but cannot be listed in a short review.
M. Henriksen (Lafayette, Ind.).

McLaughlin, J. E., and Rosenberg, Alex. Zero divisors and commutativity of rings. Proc. Amer. Math. Soc. **4**, 203–212 (1953).

An associative ring is called Zorn provided some left multiple of every non-nilpotent element is idempotent. If a Zorn ring A has all of its divisors of zero in its center C, then three possibilities arise: (1) A is commutative; (2) A is a division ring; or (3) A modulo its radical R is a field which has degree of transcendence at least 2 over $C-R$. This generalizes a theorem of I. N. Herstein [Proc. Amer. Math. Soc. **1**, 370–371 (1950); these Rev. **12**, 75] which dealt with finite rings of this type. Examples are given which demonstrate that case (3) actually arises. The above result is generalized to alternative rings. However, the identity $6(x, y, z)=((x, y), z)+((y, z), x)+((z, x), y)$ shows that for

characteristic prime to 6 only the associative case occurs. For Banach algebras the authors replace "zero divisor" by "topological zero divisor" and obtain similar results.

E. Kleinfeld (Chicago, Ill.).

Herstein, I. N. A theorem on rings. Canadian J. Math. **5**, 238–241 (1953).

If R is a (Jacobson) semi-simple ring with the property that some power of every element is in its center, then Kaplansky [same J. **3**, 290–292 (1951); these Rev. **13**, 101] has proved that R is commutative. The author extends this result to the case in which R has no nil-ideals. Kaplansky's theorem is used in the proof but otherwise the argument is self-contained, the methods being direct and ideal-theoretic. What is actually proved is the equivalent statement that the commutator ideal of R is a nil-ideal. *W. G. Lister.*

Nakayama, Tadasi. On the commutativity of certain division rings. Canadian J. Math. **5**, 242–244 (1953).

Recently a series of increasingly more general results has yielded the following: (1) If for each x in a ring R with center Z there is an integer $n(x)>1$ such that $x^{n(x)}-x \in Z$, then R is commutative [I. N. Herstein, Amer. J. Math. **75**, 105–111 (1953); these Rev. **14**, 613]; (2) If R has no nil ideals and for each x in R there is an integer $n(x)$ such that $x^{n(x)} \in Z$, then R is commutative. [See the review above.] Both of these theorems are proved by structural methods from the case R a division ring. In the present paper the author gives a condition implying commutativity in a division ring which contains both (1) and (2) as they apply to such rings. The result is that if R is a division ring whose center contains r non-zero elements $\alpha_1, \alpha_2, \cdots, \alpha_r$ such that for every element x in R there exist r positive integers $n_1(x), n_2(x), \cdots, n_r(x)$ with $n_1(x)<n_i(x)$ if $i\neq 1$ having the property $\alpha_1 x^{n_1(x)}+\alpha_2 x^{n_2(x)}+\cdots+\alpha_r x^{n_r(x)} \in Z$, then R is commutative. The critical step in the proof is a lemma on the nature of the field extensions $Z(x)$ for x in R.

W. G. Lister (Providence, R. I.).

Fuchs, L., and Szele, T. Contribution to the theory of semisimple rings. Acta Math. Acad. Sci. Hungar. **3**, 233–239 (1952). (Russian summary)

In this paper, a ring R is called semisimple if R has no non-zero nilpotent left ideals and satisfies the minimum condition on left ideals. Generalizing a classical theorem of Wedderburn and Artin, the authors show that R is semisimple if and only if every left ideal of R has a right unit, or equivalently every left ideal is generated by an idempotent. The sufficiency is proved by showing that R is a direct sum of a finite number of minimal (non-nilpotent) left ideals ~~(which are total matric algebras over skew fields, cf., e.g., Artin, Nesbitt, and Thrall, Rings with minimum condition, Univ. Michigan Publ. Math. no. 1, 1944, chapters 4 and 5; these Rev. 6, 33)~~.

They also show that every left ideal of R has a left unit if and only if R is a direct sum of a finite number of skew fields. Every subring of R has a left unit if and only if R is a direct sum of a finite number of fields F_i, each of which is an algebraic extension of the field of integers modulo some prime p_i. An interesting corollary is that every subring of a skew field F is a skew field if and only if F is an algebraic extension of the field of integers modulo some prime p. (A minor error occurs in the proof of Lemma 2; M_2 should be described as a left ideal of R which is a maximal left ideal of M_1.) *M. Henriksen* (Lafayette, Ind.).

Ikeda, Masatoshi. A characterization of quasi-Frobenius rings. Osaka Math. J. **4**, 203–209 (1952).

A quasi-Frobenius ring is a generalization of a semi-simple ring characterized originally by Nakayama and the reviewer in terms of the closure of ideals under annihilation thus: $l(r(L))=L$ and $r(l(R))=R$ for every left and right ideal L and R. It is shown here that, for rings with minimum condition possessing a unit, being quasi-Frobenius is equivalent to requiring that every operator homomorphism between two left ideals be given by a right multiplication by some element. It is shown that for algebras it is sufficient to assume this property for simple right and left ideals, but a counterexample shows that this is insufficient for rings with minimum condition. *Marshall Hall* (Columbus, Ohio).

Nagata, Masayoshi. On the theory of semi-local rings. Proc. Japan Acad. **26**, nos. 2–5, 131–140 (1950).

Let R be a commutative ring with identity, let M_n be the intersection of the nth powers of all maximal ideals, let $D(R)=\bigcap_1^\infty M_n$. If $D(R)=0$ and if M has only a finite number of maximal ideals, then R is called a semi-local ring and is topologized by taking the ideals M_n as a fundamental system of neighborhoods of zero. Most of the theorems proved by Chevalley in the Noetherian case [Ann. of Math. (2) **44**, 690–708 (1943); these Rev. **5**, 171] are carried over to the present case. It is necessary, however, to make use of a modified quotient ring $R_{[S]}=R_S/D(R_S)$, where S is a subset of R closed under multiplication, and R_S is the ordinary quotient ring in the sense of Grell-Chevalley-Uzkov. For Noetherian rings, $D(R)=(0)$, so that $R_{[S]}=R_S$.

I. S. Cohen (Cambridge, Mass.).

Nagata, Masayoshi. Note on subdirect sums of rings. Nagoya Math. J. **2**, 49–53 (1951).

Generalizing some results of the paper reviewed above, the author gives some conditions for a subdirect sum of rings to be a direct sum. There are various other results. The treatment is not confined to the commutative case.

I. S. Cohen (Cambridge, Mass.).

Nagata, Masayoshi. On the nilpotency of nil-algebras. J. Math. Soc. Japan **4**, 296–301 (1952).

Let $\mathfrak{A}$ be an algebraic algebra over a field of characteristic 0. The author proves that there is a function f on the natural numbers such that if $\mathfrak{A}$ is a nil-algebra of bounded index $\leqq n$, then $\mathfrak{A}$ is nilpotent of index $\leqq f(n)$.

A simple direct computation shows $f(2)=3$, and the method of proof consists of an inductive description of f based on a condensation of the combinatorial problem met in a direct attack. This condensation is accomplished by introducing the group algebra $\mathfrak{o}_t$, over the coefficient field, of the symmetric group S_t on t symbols and employing its elements as operators on products of t elements of $\mathfrak{A}$. If $\sigma \in S_t$ and $y_1, \cdots, y_t$ are in $\mathfrak{A}$, then $\sigma(y_1 \cdots y_t)=y_{\sigma(1)} \cdots y_{\sigma(t)}$. Roughly, elements of S_t which permute only a specified number g of elements of $1, \cdots, t$ are used to define principal left ideals of $\mathfrak{o}_t$, $\mathfrak{l}_1, \mathfrak{l}_2, \cdots, \mathfrak{l}_h$ whose sum, subject to a condition on g, is $\mathfrak{o}_t$. Let $\mathfrak{F}$ be the free algebra on t generators $u_1, \cdots, u_t$ and $\mathfrak{N}$ the ideal of $\mathfrak{F}$ generated by the nth powers of elements of $\mathfrak{F}$. Now for sufficiently large t the left ideal of $\mathfrak{o}_t$ consisting of those elements mapping $u_1 \cdots u_t$ into $\mathfrak{N}$ contains for an appropriate g generators of the $\mathfrak{l}_i$ and therefore the identity of S_t. If $y_1, \cdots, y_t$ are in $\mathfrak{A}$, the natural map determined by $u_i \rightarrow y_i$ of $\mathfrak{F}$ onto the algebra generated by $y_1, \cdots, y_t$ gives the desired result.

The notation of the paper is somewhat clumsy, and the proofs are sketchy, a good many details being left to the reader. *W. G. Lister* (Providence, R. I.).

Levitzki, Jakob. On the structure of algebraic algebras and related rings. Trans. Amer. Math. Soc. **74**, 384–409 (1953).

A ring is π-regular if for any a there exists an element x and an integer n such that $a^n x a^n = a^n$; it is an I-ring (the reviewer has used the term "Zorn ring") if every non-nil right ideal contains a non-zero idempotent. Any algebraic algebra is π-regular, and any π-regular ring is an I-ring. With the aim of making headway on the problem of local finiteness of algebraic algebras (Kurosch's problem), the reviewer has studied π-regular rings [Trans. Amer. Math. Soc. **68**, 62–75 (1950); **70**, 219–255 (1951); these Rev. **11**, 317; **13**, 48]. The author here substantially extends these results to I-rings and gives simplified purely algebraic proofs. A rather large number of related concepts are analyzed, but the main theorems are perhaps the following. 1. If A is a semi-simple I-ring which is of finite index modulo every primitive ideal, then A contains a direct summand which is a total matrix ring over a plain ring ($=I$-ring with no nilpotent elements). 2. Suppose that A and all its homomorphic images are I-rings, and that A is of finite index modulo every primitive ideal; then A possesses a well ordered ascending composition series such that every quotient is either a nil ring or a total matrix ring over a plain ring. 3. Let A be an algebraic algebra such that every primitive image of A is finite-dimensional over its center, and such that any nil ideal of any homomorphic image of A is locally finite; then A is locally finite (this corrects an error in the reviewer's second paper). Two side matters are of independent interest: the concept of the locally finite kernel of an algebraic algebra, i.e., the union of all locally finite ideals, and the invariance of the degree of the matrices in the representation of a ring as a total matrix ring over a plain ring. *I. Kaplansky* (Chicago, Ill.).

Tornheim, Leonard. On the definition of Clifford algebras. Michigan Math. J. 1 (1952), 194–197 (1953).

The author shows the equivalence of the definitions of Clifford algebras over a given field as algebras with a certain type of multiplication table for a basis and as universal algebras of certain sets of equations. *W. G. Lister.*

van Est, W. T. Some theorems on (CA) Lie algebras. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 546–557, 558–568 (1952).

The author continues the study of (CA) algebras which he inaugurated in earlier papers [same Proc. **54**, 321–328 (1951); **55**, 255–266, 267–274 (1952); these Rev. **13**, 432; **14**, 135]. A Lie algebra H is a (CA) algebra if its adjoint group (i.e., the linear Lie group generated by its inner derivations) is closed (in the full linear group). Among the principal results are the following. If H is a (CA) algebra and G a dense subalgebra of H (i.e., a subalgebra such that for some Lie group $\mathfrak{H}$ generated by H, the subgroup generated by G is dense in $\mathfrak{H}$), then any isomorphism $G \to H$ can be extended to an automorphism of H. A Lie algebra is a (CA) algebra if and only if its radical is. Every Lie algebra is the semi-direct sum of an abelian algebra and a (CA) ideal; the latter can be chosen so that its radical is a (CA) algebra with simply connected adjoint group. The final part of the paper contains theorems about simply connected solvable (CA) Lie groups $\mathfrak{G}$. The following theorem is typical: Let $\{\phi\}$ be the class of locally faithful representations of $\mathfrak{G}$ into a Lie group $\mathfrak{H}$. If $\{\phi\}$ contains a representation ϕ which is closed on the central commutator of $\mathfrak{G}$, it contains a representation ϕ' which is closed on $\mathfrak{G}$; if $\phi(\mathfrak{G})$ is dense in $\mathfrak{H}$, ϕ' can be chosen so that $\mathfrak{H}$ is the direct product of $\phi'(\mathfrak{G})$ and a central Lie subgroup. *P. A. Smith.*

Kaplansky, Irving. Dual modules over a valuation ring. I. Proc. Amer. Math. Soc. 4, 213–219 (1953).

L'auteur développe le début d'une théorie de dualité pour les modules linéairement compacts, dont la théorie élémentaire a récemment été fondée par Zelinsky [Amer. J. Math. **75**, 79–90 (1953); ces Rev. **14**, 532]. Il suppose que l'anneau des scalaires est un anneau R de valuation discrète, complet pour la topologie définie par cette valuation; si K est le corps des quotients de R, le dual algébrique d'un R-module M est défini comme le R-module des R-homomorphismes de M dans K/R (qui remplace de façon naturelle le tore de la théorie classique de Pontrjagin). Cela fait, il est facile de définir comme d'ordinaire un couple de modules en dualité, chacun devant être faiblement dense dans le dual algébrique de l'autre. En particulier, si M est un R-module linéairement compact, son dual topologique M^* est l'ensemble des homomorphismes continus dans K/R, ce dernier étant pris discret; on montre que M et M^* sont en dualité, que $M = M^{**}$ lorsque M^* est pris discret et que la topologie donnée sur M coincide avec la topologie faible définie par M^*. On a aussi bien entendu les résultats habituels sur la dualité entre sous-modules et modules quotients, et la relation entre "biorthogonalité" et adhérence faible. L'auteur montre en terminant comment ces notions et résultats ramènent plusieurs théorèmes démontrés antérieurement par Krull et Vilenkin sur les modules linéairement compacts, à des résultats connus ou facilement démontrables sur les groupes discrets. *J. Dieudonné* (Ann Arbor, Mich.).

Iseki, Kiyoshi. On the Brown-McCoy radical in topological rings. Anais Acad. Brasil. Ci. **25**, 79–86 (1953).

Let A be a topological ring, say with a unit element. Call x a G-element if the two-sided ideal it generates is all of A. Call A a G-ring if there is a neighborhood of 1 consisting of G-elements. The author proves various properties of G-rings, for instance: the set of G-elements is open; the Brown-McCoy radical is closed. Some of the results were announced earlier [C. R. Acad. Sci. Paris **234**, 1938–1939 (1952); these Rev. **13**, 815]. *I. Kaplansky* (Chicago, Ill.).

Fleischer, Isidore. Sur les corps topologiques et les valuations. C. R. Acad. Sci. Paris **236**, 1320–1322 (1953).

Let F be a topological division ring with the following property: if $B \subset F$ is bounded away from 0, then B^{-1} is bounded. The property was called "type V" by the reviewer [Duke Math. J. **14**, 527–541 (1947); these Rev. **9**, 172], and "KT_a" by Bourbaki [Eléments de mathématique, Part I, Livre III, Chap. III–IV, Actualités Sci. Ind., no. 916, Hermann, Paris, 1942, p. 57, ex. 13; these Rev. **5**, 102]. The author abolishes this class of fields by proving that (in the commutative case) the topology can be given by a valuation on an ordered (abelian) group. More generally, the same thing is true if the multiplicative commutator subgroup of F is bounded. *I. Kaplansky* (Chicago, Ill.).

Hopf, H. Einige Anwendungen der Topologie auf die Algebra. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 75–91 (1952).

A lucid expository article. *S. Eilenberg.*

Cohn, Richard M. Essential singular manifolds of difference polynomials. Ann. of Math. (2) 57, 524–530 (1953).

If P is a differential polynomial in y, with coefficients in a differential field F of characteristic zero, which vanishes for $y=0$, then whether or not $\{0\}$ is a component of the manifold of P is determined by the power products in y and its derivatives which appear with nonzero coefficients in P, the coefficients themselves being of no importance in this connection; this is clear from Ritt's low power theorem. The author shows that for difference polynomials the analogous statement is false; specifically, if F is the difference field obtained by adjoining a transformally transcendental element t to the difference field of rational numbers, then $\{0\}$ is a component of the manifold of $yy_2^2+y_1^2y_3-ty_1y_3$ but not of $yy_2^2+y_1^2y_3-y_1y_3$. That the order of these difference polynomials is 3 is no accident, for the author proves that for difference polynomials of order 2 the analogous statement is true; more precisely, $\{0\}$ is a component of the manifold of a difference polynomial P of order 2 if and only if P has a term whose weight [Cohn, Ann. of Math. (2) 53, 445–463 (1951); these Rev. 13, 103] is $<$ the weight of every other term for all values >0 of the weight parameter.

E. R. Kolchin (New York, N. Y.).

Theory of Groups

Teissier, Marianne. Sur les demi-groupes admettant l'existence du quotient d'un côté. C. R. Acad. Sci. Paris 236, 1120–1122 (1953).

The author considers demi-groups D satisfying: (1) given $a, b \in D$ there exists $x \in D$ such that $a=xb$; (2) $bx=by$ ($b, x, y \in D$) implies $x=y$; (3) D does not contain an idempotent element. In view of an example of such a demi-group given by R. Baer and F. Levi [Heidelberger Akad. Wiss. 1932, Abh. 2, 1–12, p. 7], the author calls a "demi-group of Baer and Levi" that consisting of all one-to-one mappings of an infinite set E onto subsets E' of E such that E' has the same power as E, and $E-E'$ is infinite. The author shows that every D satisfying (1), (2), (3) is isomorphic with a subdemi-group of a demi-group of Baer and Levi. A generalization is given of a demi-group of Baer and Levi, and it is shown that every D satisfying (1) and (3) can be embedded in one of these. *A. H. Clifford.*

Casas, Pablo. Introduction to the theory of groups. I, II. Revista Mat. Elem. 1, 3–11, 34–39 (1952). (Spanish)

Expository paper.

Zalceva, M. I. On the set of ordered Abelian groups. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 135–137 (1953). (Russian)

Part of problem 102 in Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., vol. 25, rev. ed., New York, 1948; these Rev. 10, 673] is to classify all (simple) orderings of a free abelian group with a finite number of generators. The author solves this problem. The question is first reduced to the archimedean case by observing that the group is a lexicographic direct sum of archimedean groups. The problem then comes to this: given two sets of rationally independent real numbers $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, when are the groups they generate order-isomorphic? The answer is that one must be able to pass from the a's to the b's by a unimodular matrix of integers, followed by multiplication by a non-zero real number. *I. Kaplansky.*

Charles, Bernard. Le centre de l'anneau des endomorphismes d'un groupe abélien primaire. C. R. Acad. Sci. Paris 236, 1122–1123 (1953).

Let R be the ring of integral endomorphisms of an abelian group G, that is, the set of all endomorphisms of the form $x\rightarrow nx$ where n is an integer. Let R' be the ring of all endomorphisms of G, let R'' be the center of R' and let R^* be the set of those endomorphisms of G for which every subgroup of G is admissible. Let $\bar{R}$ be the set of all endomorphisms which coincide with integral endomorphisms on any finite subset of G. $R\subset\bar{R}\subset R''\subset R'$ and $\bar{R}\subset R^*$. In particular, if G is a p-primary abelian group, it is proved that $\bar{R}=R^*=R''$. It might be mentioned that R^* is either the ring of p-adic integers or is a ring of integers modulo some p^n [F. W. Levi, J. Indian Math. Soc. (N.S.) 10, 29–31 (1946); these Rev. 8, 500]. Let A be the set of automorphisms of this p-primary abelian group G. The present result is related to a theorem due to Baer [Amer. J. Math. 59, 99–117 (1937)] which states that, in most cases, $A\cap R^*=A\cap R''$. *F. Haimo.*

Szele, T. On direct decompositions of Abelian groups. J. London Math. Soc. 28, 247–250 (1953).

Let an abelian torsion group A have the property that it has an element of maximum order r (in which case it is said to be bounded), and in addition that, for all a, $a \in [r/o(a)]A$ where $o(a)$ is the order of a (in which case A is said to be a regular r-bounded group). The principal result of the paper is that regular r-bounded A is a direct summand of an extension G if and only if $A\cap rG=(0)$. The proof is elementary. Corollaries are several more or less well-known results. Typical examples are: (a) A bounded torsion subgroup of a mixed abelian group is a direct summand; (b) the fundamental decomposition theorem for finite abelian groups; (c) the theorem of Prüfer [Math. Z. 17, 35–61 (1923)] that an abelian torsion group of finite rank is a direct sum of cyclic groups of prime-power order and groups of type p^∞; and (d) an abelian group with the minimal condition for subgroups is a direct sum of a finite number of groups mentioned as summands in (c) [A. Kurosch, Math. Ann. 106, 107–113 (1932)]. Further references include R. Baer [Ann. of Math. (2) 37, 766–781 (1936)].

F. Haimo (Saint Louis, Mo.).

Pickert, Günter. Nichtkommutative cartesische Gruppen. Arch. Math. 3, 335–342 (1952).

Let the group G be written additively with identity 0, without assuming it to be abelian. If there is a binary multiplication defined for elements of G, then G is said to be semi-Cartesian if the first three of the following laws hold, and Cartesian if all five hold: 1) $0a=a0=0$ for all a; 2) if $a\neq b$, and $c\neq d$, then $ca-cb\neq da-db$; 3) an element 1 exists with $1a=a1=a$ for all a; 4) for a, b, c with $a\neq b$ there exists an x such that $xa-xb=c$; 5) for a, b, c with $a\neq b$ there exists a y such that $-ay+by=c$.

A Cartesian group leads to a projective plane in which the minor Theorem of Desargues holds with axis the line at infinity and center the infinite point on the y-axis. He shows easily that if either distributive law holds, then G is abelian. The major part of the paper consists in showing that a semi-Cartesian group can be embedded in a Cartesian group. This consists essentially in adjoining a solution x for 4), defining non-associative multiplications involving x, and then taking the free sum of G and the free group generated by the new elements. Iterating this process the embedding is accomplished. This incidentally yields the construction of non-abelian Cartesian groups. *Marshall Hall.*

Zacher, Giovanni. Costruzione dei gruppi finiti a sottogruppo di Frattini identico. Rend. Sem. Mat. Univ. Padova 21, 383–394 (1952).

A result of Ore [Duke Math. J. 5, 431–460 (1939), especially pp. 444–445] is extended to show that the Frattini subgroup of a finite group is trivial if and only if the group-theoretic union N of the normal abelian subgroups of G has a complement C in G. Furthermore, let G be a finite group with a trivial Frattini subgroup, and let H be the group-theoretic union of all the normal subgroups of G included in C. The elements of C induce a set T of automorphisms in N, and $T\cong C/H$. Also, $G/H\cong L$, the holomorph of T over N. Here N turns out to be a finite abelian elementary group with a direct decomposition into factors R_i, p-groups admissible under the mappings in T with no proper subgroups so admissible. One can form a suitable semi-direct product $\bar{G}$ [Malcev, C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 87–89 (1943); these Rev. 7, 115] of N by C which is a finite group with trivial Frattini subgroup. Conversely, let N be any finite elementary abelian group decomposing into a direct product of p-groups R_i. Let T be any group of automorphisms of N for which each R_i, but no proper subgroup of R_i, is admissible. Let H be any finite group which has no non-trivial normal solvable subgroups and let C be an extension of H by T. Forming a suitable semi-direct product of N by C, we are led to a group $\bar{G}$, as above, with trivial Frattini subgroup but with at least one non-trivial normal abelian subgroup. *F. Haimo.*

Zacher, Giovanni. Caratterizzazione dei t-gruppi finiti risolubili. Ricerche Mat. 1, 287–294 (1952).

E. Best and O. Taussky [Proc. Roy. Irish Acad. Sect. A. 47, 55–62 (1942); these Rev. 4, 2] have studied groups in which the property of being a normal subgroup is hereditary: H normal in G and K normal in H implies K normal in G. They called groups of this class t-groups. The present author proves that finite soluble t-groups G of order $g=p_1^{a_1}\cdots p_k^{a_k}$ with $p_1>\cdots>p_k$ can be completely characterized by the following two properties: (i) All Sylow-subgroups of G are Abelian or Hamiltonian; (ii) G possesses a sequence of Sylow-subgroups $S_1, S_2, \cdots, S_k$ belonging to $p_1, p_2, \cdots, p_k$, respectively, such that any element s_i of S_i transforms all elements of S_j with $j<i$ into powers of themselves. *K. A. Hirsch*

Wiman, A. Über p-Gruppen von maximaler Klasse. Acta Math. 88, 317–346 (1952).

Let G be a group of order p^n where p is a prime. Let l be the class of G, i.e., the number of terms in its upper central series. Then obviously $l\leq n-1$. This paper continues the former investigations of the author on groups G of the maximal class $n-1$ [Ark. Mat. Astr. Fys. 33A, no. 6 (1946); these Rev. 8, 251]. For a group G of maximal class the upper central series is identical with the lower central series $G\supset G_2\supset\cdots\supset G_{n-1}\supset G_n=\{1\}$ inversely ordered, and so G_{n-1} is the center of G. Since the factor group G/G_2 is an abelian group of type (p, p), and each maximal subgroup of G contains the group $G'=G_2$, there exist exactly $p+1$ subgroups of index p in G. If $n>3$, then among these $p+1$ subgroups there exists exactly one which contains no element belonging to the set G_2-G_4. This subgroup G_1 is a characteristic subgroup of G. The author inserts this group G_1 in the lower central series between G and G_2. For the elements s of the set $G-G_1$ there are three possibilities. (I) $s^p=1$ holds for all s. (II) $1\neq s^p\in G_{n-1}$ holds for all s. (III) $s^p=1$ holds for some s and for the others $1\neq s^p\in G_{n-1}$. Accordingly the author says that G is a group of type I (dihedral), of type II (dicyclic), or of type III (of intermediate kind), respectively. In this way the author succeeds in generalizing his former results concerning the special case in which G_1 is abelian. *T. Szele* (Debrecen).

Kemhadze, Š. S. On the determination of regular p-groups. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 193–196 (1952). (Russian)

It is shown that the definition of regularity in finite p-groups as given by Philip Hall reduces to the simple condition that $a^pb^p=(ab)^pc^p$ for some c in the derived group of the subgroup generated by a and b, for any choice of a and b in the group. *Marshall Hall* (Columbus, Ohio).

Sanov, I. N. On a certain system of relations in periodic groups with period a power of a prime number. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 477–502 (1951). (Russian)

This paper is concerned with the Burnside subgroup $B(p^n)$ of a free group F generated by the p^nth powers of elements of F (p a prime). The results are in terms of higher commutators. Thus with sp^s-1 V's it is shown that $(U, V, \cdots, V)^{p^{n-s}}\equiv 1 \pmod{B(p^n)\cup F_{sp^s+1}}$ where F_r is the rth term in the lower central series for F. The method depends on work of Magnus [J. Reine Angew. Math. 182, 142–149 (1940); these Rev. 2, 214] who showed that if x, y are associative but not commutative indeterminates, the formal power series e^x, e^y generate a free group F. The Baker-Hausdorff formula for z in $e^xe^y=e^z$ expresses z in the form $z=x+y+\frac{1}{2}(xy-yx)\cdots$ where the remaining terms are commutator forms in x and y. In F for an arbitrary $g=e^u$ we write $u=P_0+P_1+\cdots+P_i+\cdots$ where P_i is of degree i in x. Elements with $P_0=0$ form the normal subgroup N generated by all conjugates of e^x. Those with $P_0=P_1=0$ form a normal subgroup $G_{2,x}$ which is the derived group N' of N. Calculations are carried out modulo $G_{2,x}$. *Marshall Hall.*

Szép, J. Zur Theorie der einfachen Gruppen. Acta Sci. Math. Szeged 14, 246 (1952).

If a simple group G is a product of proper subgroups H and P, $G=HP$ with $H\cap P=1$, and if the order of P is a prime number p, then the order of G is of the form $pd(1+kp)$ where $d>1$ divides $p-1$. In particular, if p is a Fermat prime, then the order of G is even. *K. A. Hirsch.*

Baer, Reinhold. Factorization of n-soluble and n-nilpotent groups. Proc. Amer. Math. Soc. 4, 15–26 (1953).

The paper deals with n-commutativity in a finite group G of order $o(G)$ and the notions derived therefrom. $x\in G$ and $y\in G$ are said to be n-commutative if $(xy)^n=x^ny^n$ and $(yx)^n=y^nx^n$ hold. G is n-abelian if all its elements are n-commutative. The elements $(ab)^n(a^nb^n)^{-1}$, for a, b in G, generate the n-commutator subgroup which is the meet of all the normal subgroups with n-abelian factor groups. The n-centre of G is the characteristic subgroup consisting of all the elements z in G satisfying $(zg)^n=z^ng^n$ and $(gz)^n=g^nz^n$ for every g in G. The notions of n-derived series, n-soluble, and n-nilpotent are defined accordingly. If G is n-abelian [n-soluble, resp., n-nilpotent], it is also $(1-n)$-abelian [$(1-n)$-soluble, resp., $(1-n)$-nilpotent]. If every prime factor of $o(g)$ is a factor of [is relatively prime to] n, then g belongs to the n-component G_n [to the P_n-component G_{P_n}] of G. These components may not be groups. If, however, G is n-abelian, G_n, $G_{(1-n)}$ and $G_{P_n(1-n)}$ are normal subgroups (the last one is abelian) and G is their direct product (theorem A). If G is n-soluble, $o(G)=hk$, h and k are rela-

tively prime, and n as well as $1-n$ is prime to at least one of the numbers h and k, then there exist subgroups U and V, $o(U)=h$, $o(V)=k$, such that every $g \in G$ can be represented in one and only one way by $g=uv$, $u \in U$, $v \in V$ (theorem B). If G is n-nilpotent, $G=G_n\times G_{(1-n)}\times G_{Pn(1-n)}$, the third direct factor being nilpotent (theorem C). In an appendix, the author sketches proofs of theorems on infinite groups which correspond to the theorems A and C. The extension of theorem B to infinite groups is an unsolved problem since the proof depends essentially on Schur's splitting theorem.

F. W. Levi (Berlin).

Magnus, W. Errata: A connection between the Baker-Hausdorff formula and a problem of Burnside. Ann. of Math. (2) **57**, 606 (1953).

See same Ann. (2) **52**, 111–126 (1950); these Rev. **12**, 476.

Stender, P. V. On the application of the sieve method to the solution of the word problem for certain groups with a denumerable set of generating elements and a denumerable set of defining relations. Mat. Sbornik N.S. **32(74)**, 97–108 (1953). (Russian)

The author proves that Tartakovskiĭ's sieve method for the solution of the word problem in certain classes of groups [these Rev. **11**, 493; **13**, 528] can be extended to groups with a countable number of generators and of defining relations provided that the following two additional requirements are satisfied: (1) every generator occurs only in a finite number of basic words, i.e., left-hand sides of defining relations; (2) the lengths of the left-hand sides of the defining relations are uniformly bounded. The original proofs can be adapted with small modifications to the present situation.

K. A. Hirsch (London).

Scott, W. R. Transitive sets of homomorphisms. Proc. Amer. Math. Soc. **4**, 175–177 (1953).

The following statements about groups G and H whose orders are greater than n are proved equivalent: (1) If $a_1, \cdots, a_n$ are distinct elements of G and $x_1, \cdots, x_n$ are distinct elements of H, none being the identity, then there is a homomorphism mapping each a_i into the corresponding x_i. (2) If $b_1, \cdots, b_{n+1}$ are distinct elements of G and $y_1, \cdots, y_{n+1}$ are distinct elements of H, then there is an element h of H and a homomorphism σ on G into H such that $h(b_i\sigma)=y_i$ for each i. The statements are untrue if $n>3$. Necessary and sufficient conditions for their truth when n is 2 or 3 are given, and sufficient conditions for their truth when n is 1. The proofs are, as the author says, all trivial.

H. A. Thurston (Bristol).

Fuchs, L. Rédeian skew product of operator groups. Acta Sci. Math. Szeged **14**, 228–238 (1952).

For the definition and properties of the "skew product" see Rédei [Comment. Math. Helv. **20**, 225–264 (1947); these Rev. **9**, 131]. In a recent paper [J. Reine Angew. Math. **188**, 201–227 (1950); these Rev. **14**, 13] Rédei has given a set of twenty-two conditions which are necessary and sufficient for the pairs of elements of two groups to form a skew product with a suitable definition of composition. In the present papers the author shows that if the two groups have one and the same domain of operators and if their skew product is to admit the same domain of operators, then a further set of fourteen conditions is necessary and sufficient. These conditions are concerned with the effect of the operators on the elements of the two groups and on the pairs which constitute the elements of the skew product. Specialisation leads to the Schreier extension theory for groups with operators and to other known formations of operator groups from two given operator groups.

K. A. Hirsch (London).

Farahat, H. On p-quotients and star diagrams of the symmetric group. Proc. Cambridge Philos. Soc. **49**, 157–160 (1953).

The equivalence of the Robinson-Staal [Robinson, Amer. J. Math. **70**, 277–294 (1947); these Rev. **10**, 678; Staal, Canadian J. Math. **2**, 79–92 (1950); these Rev. **11**, 415] theory of star diagrams and Littlewood's [Proc. Roy. Soc. London. Ser. A. **209**, 333–353 (1951); these Rev. **14**, 243] theory of p-quotients, in the representation theory of symmetric groups, is discussed. To Robinson's [Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 20–25 (1947); these Rev. **10**, 678] formula for the p-exponent in the degree of an irreducible representation is given a direct proof, which avoids the use of the reviewer's [Jap. J. Math. **17**, 165–184 (1941); these Rev. **3**, 195; **4**, 340] one. *T. Nakayama.*

Motzkin, T. S., and Taussky, Olga. On representations of finite groups. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 511–512 (1952).

If every pair of matrices A, B of a faithful representation (by matrices over the complex field) of a finite group Γ has the property (L) that their roots α_i and β_i can be ordered so that the roots of $\lambda A+\mu B$ are $\lambda\alpha_i+\mu\beta_i$ for all complex λ and μ, then Γ is abelian; also, if the roots of $\prod C_k$, each $C_k=A$ or B, are the corresponding products of the α_i and β_i, Γ is abelian. If the matrix elements may be in a field of finite characteristic, examples show that $p(A, B)$ may have roots $p(\alpha_i, \beta_i)$ for every polynomial and yet Γ be non-abelian.

W. Givens (Knoxville, Tenn.).

Hattori, Akira. On the multiplicative group of simple algebras and orthogonal groups of three dimensions. J. Math. Soc. Japan **4**, 205–217 (1952).

Poursuivant ses recherches sur la structure du groupe multiplicatif d'une algèbre simple [Jap. J. Math. **21**, 121–129 (1952); ces Rev. **14**, 529] l'auteur démontre d'abord le résultat suivant: si B est une sous-algèbre simple d'une algèbre simple A (de rang fini sur son centre F), contenant F et distincte de A, et si $[B]$ désigne l'ensemble des sous-algèbres simples F-isomorphes à B, la représentation du groupe $I(A)$ des automorphismes intérieurs de A, comme groupe de permutations de l'ensemble $[B]$, est fidèle (ce qui entraîne que l'ensemble $[B]$ est infini). Dans la seconde partie du mémoire, l'auteur démontre que le groupe $O_3^+(Q, f)$ des rotations de l'espace à trois dimensions sur le corps des rationnels, pour une forme f d'indice 0, admet toujours une suite infinie descendante de sous-groupes distingués; ce résultat a déjà été publié par le rapporteur, avec une démonstration très analogue [C. R. Acad. Sci. Paris **233**, 541–543 (1951); ces Rev. **13**, 205]. Enfin, dans la troisième partie, l'auteur détermine la structure des algèbres non commutatives A, ayant un élément unité telles qu'une forme quadratique non dégénérée soit définie sur A, que le centre Z de A ne soit pas isotrope pour cette forme, et que, si B est le sous-espace orthogonal à Z, toute rotation dans B soit déterminée par un automorphisme intérieur de A. Il montre que A est alors nécessairement composée directe d'une algèbre de quaternions généralisée et d'une algèbre commutative ayant un élément unité.

J. Dieudonné (Evanston, Ill.).

Lazard, Michel. Détermination et généralisation des groupes de dimension des groupes libres. C. R. Acad. Sci. Paris **236**, 1222–1224 (1953).

A descending sequence $(H_i)_{i=1,2,\ldots}$ of subgroups of a group G is called an N-sequence if, for any i and j, the commutator of an element of H_i and an element of H_j lies in H_{i+j}. Such sequences may be obtained as follows. Let h be a homomorphism of G into the group of invertible elements of an associative algebra which is filtered, the filtration v being such that $v(h(x)-1)\geqq 1$ for all x in G. If H_i is the set of $x \in G$ such that $v(h(x)-1)\geqq i$, then (H_i) is an N-sequence. This is applied to the following case: A is the algebra of formal noncommutative power series in certain indeterminates x_i $(i \in I)$ over the ring of p-adic integers, p being a prime, and G is the free group generated by the elements $1+x_i$. Set $w(y)=k$ if $y \in p^kA$, y non-$\in p^{k+1}A$; then w is a filtration, whose corresponding sequence (G_i) is the central descending sequence. By using other filtrations satisfying certain conditions, the author defines new N-sequences, whose terms he expresses in terms of the groups G_i and of a certain arithmetical function depending on the filtration. Among the sequences obtained in this manner there occur in particular the sequences of dimension groups of G. Each N-sequence in G defines a topology in G, in which the groups of the sequence form a fundamental system of neighbourhoods of 1. Under certain conditions, the topology associated to the sequence defined by a filtration of A is independent of this filtration; in this manner, a certain topology, the p-topology, is defined on G. This topology is precompact. The topologies defined by distinct primes p are "independent" of each other (in the same sense as the valuations of a field are). *C. Chevalley* (New York, N. Y.).

Hu, Sze-Tsen. On local structure of finite-dimensional groups. Trans. Amer. Math. Soc. **73**, 383–400 (1952).

As a step toward the problem whether a locally compact separable connected n-dimensional group G is locally the direct product of a compact 0-dimensional group and a Lie group germ, Montgomery [Ann. of Math. (2) **52**, 591–605 (1950); these Rev. **12**, 673] proved that G is locally the topological product of a 0-dimensional subset and a connected locally connected n-dimensional local normal divisor. In the case that the center of G is locally connected, the author proves that G is even the local group-theoretical and topological (but not direct) product of a Cantor subgroup and a connected locally connected n-dimensional local normal divisor. *H. Freudenthal* (Utrecht).

Areškin, G. Ya. Operator lattices of locally compact topological groups with a countable basis. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 67–91 (1951). (Russian. Georgian summary)

A proof is given for a theorem announced earlier [Doklady Akad. Nauk SSSR (N.S.) **81**, 129–132 (1951); these Rev. **13**, 534], giving an axiomatic characterization of operator lattices which can serve as bases for closed sets in a locally compact group with a countable open basis. An analogous theorem is stated and proved for compact metric groups, and conditions, in terms of these lattices, under which a given group is isomorphic or homomorphic to another group are also presented. *E. Hewitt* (Seattle, Wash.).

Wallace, A. D. A note on mobs. Anais Acad. Brasil. Ci. **24**, 329–334 (1952).

A mob S is a topological (Hausdorff) semi-group, that is, S is a Hausdorff space and a semi-group in which the mapping $(x, y)\rightarrow xy$ is continuous. Among the theorems proved are: If A is an algebraic subgroup whose closure $\bar{A}$ is compact, then $\bar{A}$ is a topological subgroup. (There is an obvious misprint at this point.) This extends a result of Iwasawa [Sûgaku **1**, 94–95 (1948)] and Gelbaum, Kalisch, and Olmsted [Gelbaum, Kalisch, and Olmsted, Proc. Amer. Math. Soc. **2**, 807–821 (1951); these Rev. **13**, 206]. In the terminology of Rees [Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127] a compact mob has a minimal closed ideal T with a unit element. (This was shown independently by Numakura [Math. J. Okayama Univ. **1**, 99–108 (1952); these Rev. **14**, 18] who gives many additional properties of T.) If S is connected and has a unit, then any minimal closed ideal is connected. If S is compact and has a unit, the set of elements with a two-sided inverse form a topological subgroup. If S is compact and Abelian and if every element is idempotent, then S has a zero element. Some additional theorems on connectivity are given. Examples are given which show the necessity of some of the above sufficient conditions. *M. Henriksen.*

NUMBER THEORY

Thébault, Victor. Questions d'arithmétique. Mathesis **62**, 14–20 (1953).

Moessner, Alfred. A property of Pythagorean numbers. Riveon Lematematika **6**, 27 (1953). (Hebrew)

Grün, Otto. Über ungerade vollkommene Zahlen. Math. Z. **55**, 353–354 (1952).

If an odd perfect number $m=p_1^{\alpha_1}\cdots p_n^{\alpha_n}$ exists which has n different prime divisors, then its least prime divisor p_1 is less than or equal to n. This theorem was proved by C. Servais [Mathesis **8**, 92–93 (1888)]. The author improves this result by proving that $p_1<2n/3+2$. *A. Brauer.*

Kanold, Hans-Joachim. Einige Bemerkungen über befreundete Zahlen. Arch. Math. **3**, 282–284 (1952).

The author shows that if $m=2^np^{2a}$ and $m'=q^{2\beta}$ or $q^{2\beta}r^{2\gamma}$, where p, q, r are odd primes, then the pair (m, m') is not amicable. *D. H. Lehmer* (Los Angeles, Calif.).

***Brun, Viggo, Stubban, J. O., Fjeldstad, J. E., Tambs Lyche, R., Aubert, K. E., Ljunggren, W., and Jacobsthal, E. On the divisibility of the difference between two binomial coefficients.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 42–54. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

This joint paper relates to a theorem of Brun [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 80–88] to the effect that the difference

$$(1)\qquad \binom{2^{n+1}}{2^n}-\binom{2^n}{2^{n-1}}$$

is divisible by 2^{3n}. This was discovered indirectly in studying the differential equation $y'=y^2+1$ and an interesting reconstruction of this discovery is given by Brun. A short proof of the theorem is given by Stubban using the theorem on the sum of the squares of the binomial coefficients. Fjeldstad strengthens Brun's result by showing that (1) is divisible by

2^{3n} for $n \geqq 2$. The proof begins with Stubban's method and makes a more elaborate analysis of the quotient of (1) on division by 2^{3n+1}. Tambs Lyche mentions a result generalizing Brun's theorem. This has to do with the highest power of 2 dividing the elementary symmetric functions of the n first odd integers.

Aubert points out that in general the difference

$$\binom{2^m}{2^{m-1}} - \binom{2^n}{2^{n-1}}$$

is divisible by 2^{3n} when $m \geqq n > 1$, the quotient being odd. More generally, the difference between the kth powers of these binomial coefficients is exactly divisible by $2^{3n+k+a-1}$. Here 2^a is the highest power of 2 dividing k. Ljunggren proves that if we replace the number 2 in (1) by a prime p, then the resulting difference is divisible by $p^{3n+\epsilon}$ where $\epsilon = 0$, 1 or 2 according as $p=2$, $p=3$, or $p>3$. Jacobsthal generalizes all the above by giving a complicated congruence involving Bernoulli numbers, for differences of the form

$$\binom{pm}{pn} - \binom{m}{n}.$$

D. H. Lehmer (Los Angeles, Calif.).

Vandiver, H. S. A supplementary note to a 1946 article on Fermat's last theorem. Amer. Math. Monthly **60**, 164–167 (1953).

Additions to the bibliographical references to papers on the first case of Fermat's last theorem, given in a former article [same Monthly **53**, 555–578 (1946); these Rev. **8**, 313] by the same author. *N. G. W. H. Beeger.*

Reissig, Rolf. Die pandiagonalen Quadrate vierter Ordnung. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **100**, no. 6, 54 pp. (1952).

Generalizing the procedure of W. Schnee [same Ber. **98**, no. 1 (1951); these Rev. **13**, 322], the author makes a thorough investigation of magic squares whose entries are not necessarily consecutive. He remarks that, for a pandiagonal square, the $4n$ linear equations connecting the n^2 entries form a system whose rank is $4n-4$ or $4n-3$ according as n is even or odd. Turning to the special case $n=4$, he calls a square "normal" when the sixteen entries a_{ik} (all distinct) are adjusted so that $a_{11}=1$ and $a_{12}<a_{21}<a_{41}<a_{14}$. Using a representation by lattice points, he obtains a formula for $f(s)$, the number of normal pandiagonal squares of the fourth order having a given even number s for the sum of the entries in each row or column or diagonal; e.g., $f(s)=0$ for $s \leqq 32$, $f(34)=f(36)=1$, $f(38)=3$, $f(40)=2$, $f(42)=7$. *H. S. M. Coxeter* (Toronto, Ont.).

Moessner, Alfred. A Diophantine problem. Riveon Lematematika **6**, 26–27 (1953). (Hebrew)

Moessner, Alfredo. Due problemi diofantei. Boll. Un. Mat. Ital. (3) **8**, 71–73 (1953).

Moessner, Alfred. On the equation $x^m = py^m + (-1)^m$, $m=1, 2$. Euclides, Madrid **13**, 119 (1953). (Spanish)

Barnes, E. S. On the Diophantine equation $x^2+y^2+c=xyz$. J. London Math. Soc. **28**, 242–244 (1953).

A solution $x=u_0$, $y=u_1$, $z=v$ of the equation of the title is called fundamental if $u_0 \leqq u_1$, and $0<u_0\leqq c$ or $0 \leqq u_0 \leqq \sqrt{-c}$ according as c is positive or negative, all solutions being subject to the condition $(xy, c)=1$. It is proved that any solution with $0 \leqq x \leqq y$ is either a fundamental solution or generated therefrom by the recurrence relation $u_{r+1}=vu_r-u_{r-1}$; thus $x=u_r$, $y=u_{r+1}$, $z=v$ for some positive integer r. The proof is by descent, similar to one of L. J. Mordell [Acta Math. **88**, 77–83 (1952); these Rev. **14**, 536]. *I. Niven.*

Morgantini, Edmondo. Sulla risoluzione dell'equazione diofantea: $\sum_i a_i x_i^2 = x_0^{2m} \sum_i a_i$. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 93–101 (1952).

This is substantially a special case of the treatment in the paper next reviewed. *J. W. S. Cassels.*

Morgantini, Edmondo. Sulla risoluzione dell'equazione diofantea: $\sum_i a_i x_i^2 = \sum_i a_i y_i^{2m_i}$. Ann. Triestini. Sez. 2. (4) **5**(21) (1951), 35–45 (1952).

A parametrisation of the title equation where the a_i are rational integers and the m_i are natural numbers in terms of variables y_i, α_i is obtained by putting $x_i + y_i^{m_i} = \alpha_i t$ and solving the resultant linear equation for t. The problem of finding all integer solutions thus leads to a multiplicative diophantine equation [Skolem, Diophantische Gleichungen, Springer, Berlin, 1938, Kap. IV]. *J. W. S. Cassels.*

Morgantini, Edmondo. Sulla rappresentazione parametrica di un'ampia classe di varietà unirazionali e sulle sue applicazioni all'analisi diofantea. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 238–267 (1952).

The author considers integer solutions of the diophantine equation

$$\sum_{i=1}^{r} a_i x_i^{\rho m_i} = \sum_{i=1}^{s} b_i y_i^{\sigma n_i}$$

where the a_i, b_i are polynomials with integer coefficients in variables $z_1, \cdots, z_t$ and ρ, m_i, σ, n_i are natural numbers [cf. Morgantini, Rend. Sem. Mat. Univ. Padova **21**, 44–57 (1952); these Rev. **14**, 247]. On putting $x_i = A_i x^{\mu_i}$, $y_i = B_i y^{\nu_i}$, where the μ_i, ν_i are chosen so that $m_1\mu_1 = m_2\mu_2 = \cdots = m$, $n_1\nu_1 = n_2\nu_2 = \cdots = n$, and the A_i, B_i, z_i are taken as parameters, there results an equation of the type

$$U(A, B, z)x^{m\rho} = V(A, B, z)y^{n\sigma};$$

which gives a parametrisation of the surface if $m\rho$ is prime to $n\sigma$. Integer solutions are discussed. As with the two preceding papers, the discussion is cloaked in an impressive geometrical terminology. *J. W. S. Cassels.*

Heegner, Kurt. Diophantische Analysis und Modulfunktionen. Math. Z. **56**, 227–253 (1952).

The author applies the transformation theory of modular forms for the case of singular moduli (complex multiplication) to obtain results on the solutions of various cubic and quartic diophantine equations in two variables. The solutions are usually in the rational field or the class-field of the multiplication. Most of the equations considered are of the form $p(au^4+b)$ or $p(cu^3+d)$ equal to a square. Here p is a prime and a, b, c, d are small integers fixed by the complex multiplication being considered. The author proves in addition, by extending certain results of Weber on complex multiplication, that the only quadratic fields with negative discriminant and class number unity are the known classical cases. *M. Ward* (Pasadena, Calif.).

Cohen, Eckford. Representations by cubic congruences. Proc. Nat. Acad. Sci. U. S. A. **39**, 119–121 (1953).

In this paper the number $N_s(\alpha, p)$ of solutions of the congruence $\alpha \equiv x_1^3 + \cdots + x_s^3 \bmod p$ is considered when p is

a prime, $p\equiv 1 \bmod 3$. It is shown that $N_s(\alpha, p)$ can be computed by help of Gauss sums in the following special cases: 1) $s=3$, α cubic residue mod p; 2) $s=3$, $\alpha=0$.

H. Bergström (Gothenburg).

Carlitz, L. Congruences connected with three-line latin rectangles. Proc. Amer. Math. Soc. 4, 9–11 (1953).

Write $\Delta f(n)=f(n+m)-2^m f(n)$, $\Delta^r f(n)=\Delta\Delta^{r-1}f(n)$. Let K_n be the number of reduced three-line Latin rectangles. Riordan [Amer. Math. Monthly 59, 159–162 (1952); these Rev. 13, 813] proved that

$$K_n=n^2K_{n-1}+(n)_2K_{n-2}+2(n)_3K_{n-3}+k_n,$$

where k_n satisfies the recurrence relation

$$k_n+nk_{n-1}=-(n-1)2^n.$$

The author proves $\Delta^r k_n\equiv\Delta^r K_n\equiv 0 \ (m^r)$. This generalizes the congruence (14) of Riordan's paper. *H. B. Mann.*

Carlitz, L. Note on a theorem of Glaisher. J. London Math. Soc. 28, 245–246 (1953).

Put

$$(x+1)(x+2)\cdots(x+p-1)=x^{p-1}+A_1x^{p-2}+\cdots+A_{p-1}.$$

Glaisher [Quart. J. Pure Appl. Math. 31, 1–35 (1899)] proved that if r is odd and >3 and p is a prime >5, then $\frac{1}{2}p(p-r)A_{r-1}-A_r\equiv 0 \pmod{p^4}$. In the present note the author establishes the sharper congruence:
$\frac{1}{2}p(p-r)A_{r-1}-A_r\equiv r(r-1)(r-2)B_{r-3}p^4/24(r-3) \pmod{p^5}$, where B_{r-3} denotes the Bernoulli number in the even suffix notation. An analogous result is obtained for the numbers $\bar{A}_r=(p!)^{-1}\sum_{s=0}^{p}(-1)^{p-s}\binom{p}{s}s^{p+r}$, which have properties similar to those of A_r. The proofs are based upon some additional formulas of Glaisher [loc. cit.].

A. L. Whiteman (Princeton, N. J.).

Rai, T. On a problem of additive theory of numbers. IV. J. Sci. Res. Banaras Hindu Univ. 2, 219–220 (1952).

$W(k, s)$ is the least value of j for which

$$\sum_{i=1}^{j}x_{i_1}^{l}=\cdots=\sum_{i=1}^{j}x_{i_s}^{l},\quad 1\leq l\leq k,$$

have a nontrivial solution such that no two of the sums $\sum$ are equal. Application of a known theorem to known (numerical) solutions yields new solutions from which follow $W(4, 3)\leq 7$ and $W(6, 3)\leq 12$.

N. G. W. H. Beeger (Amsterdam).

Moser, Leo. On non-averaging sets of integers. Canadian J. Math. 5, 245–252 (1953).

Denote by $\nu(n)$ the maximum number of integers not exceeding n so that no three of them form an arithmetic progression. The author proves that $\nu(n)<4n/11+5$. Roth recently proved that $\nu(n)=o(n)$ [C. R. Acad. Sci. Paris 234, 388–390 (1952); these Rev. 13, 724]. The author further constructs an infinite sequence of integers $a_1<a_2<\cdots$ no three of which form an arithmetic progression and so that the number of a's not exceeding n is greater than $n^{1-c/(\log n)^{\frac{1}{2}}}$. Behrend previously showed that $\nu(n)>n^{1-c/(\log n)^{\frac{1}{2}}}$ [Proc. Nat. Acad. Sci. U. S. A. 32, 331–332 (1946); these Rev. 8, 317], but Behrend's sequence depended on n; thus he did not get an infinite sequence with the above property.

P. Erdös (Los Angeles, Calif.).

Prachar, K. On the sum $\sum_{p\leq x}\omega(f(p))$. J. London Math. Soc. 28, 236–239 (1953).

Let $v(n)$ denote the number of distinct prime divisors of n. The author proves that $\sum_{p\leq x}v(f(p))>cx\log\log x/\log x$ if $f(x)$ is an irreducible polynomial with integral coefficients $\neq ax$. The proof depends upon a theorem of Titchmarsh concerning an estimate for $\pi(x, a, k)$, the number of primes in the arithmetic progression $k+an$, $1\leq n\leq x$, where a may depend upon x. *R. Bellman* (Santa Monica, Calif.).

Erdös, P. On a conjecture of Hammersley. J. London Math. Soc. 28, 232–236 (1953).

Denote by $\Sigma_{n,s}$ the sth elementary symmetric function of $1, 2, \cdots, n$. Hammersley [Proc. London Math. Soc. (3) 1, 435–452 (1951); these Rev. 13, 725] conjectured that the value of s which maximizes $\Sigma_{n,s}$ for a given n is unique.

The author proves this conjecture, as well as the result that for sufficiently large n all the integers $\Sigma_{n,s}$, $1\leq s\leq n$, are different. He states that Stone and he have proved the following: Let $u_1<u_2<\cdots$ be an infinite sequence of positive reals such that $\sum u_i^{-1}=\infty$ and $\sum u_i^{-2}<\infty$. Denote by $\sum_{n,s}$ the sum of the products of the first n of them taken s at a time, and denote by $f(n)$ the largest value of s for which $\Sigma_{n,s}$ assumes its maximum value. Then

$$f(n)=n-\left[\sum_{i=1}^{n}u_i^{-1}-\sum_{i=1}^{\infty}u_i^{-2}(1+u_i^{-1})^{-1}+o(1)\right].$$

N. J. Fine (Philadelphia, Pa.).

Basu, N. M. A note on partitions. Bull. Calcutta Math. Soc. 44, 27–30; corrections, 142 (1952).

Let $B_n^{(s)}$ denote the number of compositions (order of parts taken into account) of n into parts not exceeding s. By use of the generating function $(1-z-z^2-\cdots-z^s)^{-1}$, the author shows that there exists a unique $\alpha=\alpha(s)$, $\frac{1}{2}<\alpha<1$, and an explicitly given $K=K(\alpha, s)$ such that $B_n^{(s)}\sim K/\alpha^{n-1}$, as $n\to\infty$. *N. J. Fine* (Philadelphia, Pa.).

Auluck, F. C. On partitions of bipartite numbers. Proc. Cambridge Philos. Soc. 49, 72–83 (1953).

Let $p(m, n)$ be the number of partitions of the bipartite number (m, n). The author finds asymptotic expressions for $p(m, n)$ in the two cases: (a) m is fixed; (b) m and n are of the same order. *N. J. Fine* (Philadelphia, Pa.).

Horváth, J. Prime numbers. I, II, III. Revista Mat. Elem. 1, 21–33, 70–78 (1952) 2, 21–37 (1953). (Spanish)

Expository paper.

✱Selberg, Atle. On elementary methods in primenumber-theory and their limitations. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 13–22. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

This is a review of methods based on the general idea of the sieve of Eratosthenes. The well-known form given to it by Viggo Brun is developed into a more flexible instrument by the removal of certain restrictions. Some indications of the procedure are given, but in the main the lecture is a descriptive account without detailed proofs. The more general method leads to improvements in earlier results; thus it is stated that it can be proved that every sufficiently large even integer can be expressed as a sum of two positive integers each having at most three prime factors. The method is, however, subject to natural limitations, and the author develops the thesis that such methods alone cannot

be expected to solve the deeper problems of the theory of primes. At the same time he discusses the possibility of going further by widening the concept of "sieve", and indicates how this idea led him to the asymptotic formula underlying elementary proofs of the prime number theorem.

A. E. Ingham (Cambridge, England).

Salem, Raphaël. Sur une proposition équivalente à l'hypothèse de Riemann. C. R. Acad. Sci. Paris **236**, 1127–1128 (1953).

A necessary and sufficient condition for the Riemann hypothesis, derived from a theorem of Wiener on Fourier transforms, is that the integral equation

$$\int_{-\infty}^{\infty} \frac{e^{-\sigma y}\phi(y)dy}{e^{e^{x-y}}+1}=0$$

should have no bounded solution $\phi(y)$ other than the trivial solution $\phi(y)=0$, for $\frac{1}{2}<\sigma<1$. *E. C. Titchmarsh.*

Ricci, Giovanni. La differenza di numeri primi consecutivi. Univ. e Politecnico Torino. Rend. Sem. Mat. **11**, 149–200 (1952).

The author reviews the history of the various problems concerned with the difference between consecutive primes, including the latest results on the values of α in the inequality $p_{n+1}-p_n<p_n^{\alpha}$. It is also shown, for example, that the inequality $p_{n+1}-p_n>(e^C-\epsilon)\log p_n$, where $\epsilon>0$ and C is Euler's constant, holds for an infinity of n. Various data on the differences between successive primes <1000 are given in tabular and graphical form. *D. H. Lehmer.*

Prachar, K. Über Primzahldifferenzen. Monatsh. Math. **56**, 304–306 (1952).

The author proves, by using Brun's method, that the density of integers of the form $p_{k+1}-p_k$ (p_k the sequence of primes) is positive. *P. Erdös* (Los Angeles, Calif.).

Prachar, K. Über Primzahldifferenzen. II. Monatsh. Math. **56**, 307–312 (1952).

Let $q_1<q_2<\cdots$ be the sequence of consecutive primes $\equiv l \pmod{D}$, $(l, D)=1$. The author proves that the density of integers of the form $q_{k+1}-q_k$ is positive; also that $\liminf (q_{k+1}-q_k)/\log q_k<\phi(D)$; for $D=1$ this is a result of the reviewer [Duke Math. J. **6**, 438–441 (1940); these Rev. **1**, 292].

Denote by $\Delta^k a_n$ the kth difference of the sequence a_n. The author proves that $\sum_{q_n<x} \Delta^k(q_{n+1}-q_n)/q_n>c\log x$; for $k=1$, $D=1$ this was proved by the reviewer and Rényi [Simon Stevin **27**, 115–125 (1950); these Rev. **11**, 644].

P. Erdös (Los Angeles, Calif.).

Weil, André. Sur les "formules explicites" de la théorie des nombres premiers. Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.] Tome Supplémentaire, 252–265 (1952).

The author gives a far-reaching generalization (sums extended over the zeros of the zeta functions of Hecke "mit Grössencharakteren") of the explicit formulae in the theory of prime numbers. The "Grössencharaktere" χ are representations of the group of idèle-classes of an algebraic number field k into the multiplicative group of non-zero complex numbers. An idèle $a=(a_v)$ is an element of the group $\prod_v k_v^*$, where the k_v^* are the multiplicative groups of the non-zero elements of the completions k_v of k with respect to its valuations v. For the elements of the subgroup $\prod_\lambda k_\lambda^*$, where λ runs through the r_1 real Archimedean valuations and the r_2 complex Archimedean valuations of k, the character χ has the form

$$\chi(a_1, \cdots, a_{r_1+r_2})=\prod_{\lambda=1}^{r_1+r_2}\left(\frac{a_\lambda}{|a_\lambda|}\right)^{-f_\lambda}|a_\lambda|^{i\eta_\lambda\varphi_\lambda},$$

where the φ_λ are real numbers and the f_λ are integers; for the real valuations we have $\eta_\lambda=1$ and $f_\lambda=0$ or $=1$; for the complex valuations we have $\eta_\lambda=2$. Let $\Phi(s)$ be the transform $\int_{-\infty}^{+\infty} F(x)e^{(s-\frac{1}{2})x}dx$ of a complex-valued function $F(x)$ and let $\omega=\beta+i\gamma$ run through the zeros in the critical strip $0\leqq\beta\leqq1$ of the zeta function of Hecke:

$$L(s)=\sum_{\mathfrak{a}}\chi(\mathfrak{a})\cdot N\mathfrak{a}^{-s}=\prod_{\mathfrak{p}}(1-\chi(\mathfrak{p})\cdot N\mathfrak{p}^{-s})^{-1}$$

($\mathfrak{a}$ runs through all integral ideals of k, prime to the conductor $\mathfrak{f}$ of χ; $\mathfrak{p}$ runs through all prime ideals not dividing $\mathfrak{f}$). The author proves that (if suitable conditions are imposed on $F(x)$):

$$\text{(I)}\quad \lim_{T\to+\infty}\sum_{|\gamma|<T}\Phi(\omega)=\delta_\chi\int_{-\infty}^{+\infty}F(x)(e^{\frac{1}{2}x}+e^{-\frac{1}{2}x})dx+F(0)\log A_+$$
$$-\sum_{\mathfrak{p}}\sum_{n=1}^{\infty}\frac{\log N\mathfrak{p}}{N\mathfrak{p}^{\frac{1}{2}n}}[\chi(\mathfrak{p})^nF(\log N\mathfrak{p}^n)+\chi(\mathfrak{p})^{-n}F(\log N\mathfrak{p}^{-n})]$$
$$-\sum_{\lambda=1}^{r_1+r_2}\mathrm{PF}\int_{-\infty}^{+\infty}F(x)e^{-i\varphi_\lambda x}K_{\eta_\lambda, f_\lambda}(x)dx.$$

Here $\delta_\chi=1$ for the identical character and $=0$ otherwise; $A=(2\pi)^{-d}|\Delta|N\mathfrak{f}$, where d is the degree and Δ the discriminant of k;

$$K_{1,f}(x)=e^{(\frac{1}{2}-f)|x|}|e^x-e^{-x}|^{-1},\quad K_{2,f}(x)=e^{-\frac{1}{2}f|x|}|e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}|^{-1}$$

and the symbol PF is defined by

$$\mathrm{PF}\int_{-\infty}^{+\infty}\alpha(x)dx=\lim_{\lambda\to\infty}\left[\int_{-\infty}^{+\infty}(1-e^{-\lambda|x|})\alpha(x)dx-2\beta(0)\log\lambda\right],$$

where $\beta(x)=|x|\alpha(x)$. The necessity of defining this symbol arises from the fact that an application of the Fourier integral theorem for the two functions $\Psi(t)=\Phi(\frac{1}{2}+it-i\varphi_\lambda)$ (Fourier transform of $F(x)e^{-i\varphi_\lambda x}$) and $\Re[\Gamma'/\Gamma(\frac{1}{2}+it)]$ introduces the Fourier transform of the latter function and this transform is not a function but a distribution $-\pi\mathrm{PF}(|e^{\frac{1}{2}x}-e^{-\frac{1}{2}x}|^{-1})$ in the sense of L. Schwartz, the distribution $\mathrm{PF}\alpha$ being defined by $\mathrm{PF}\alpha(\varphi)=\mathrm{PF}\int_{-\infty}^{+\infty}\alpha(x)\varphi(x)dx$. The author further proves that a necessary and sufficient condition for the validity of the Riemann hypothesis for $L(s)$ is that the right-hand side of (I) is $\geqq0$ for all functions $F(x)$ of a certain class. He also gives a necessary and sufficient condition for the validity of the Riemann hypothesis for all functions $L(s)$ belonging to k and this in the form that a certain distribution on the group of idèle-classes should be of positive type. *H. D. Kloosterman.*

Singh, Daljit. The numbers $L(m, n)$ and their relations with prepared Bernoullian and Eulerian numbers. Math. Student **20**, 66–70 (1952).

The author expands s^r as a binomial series

$$s^r=\sum_{k=1}^{r}L(r, k)\binom{s+k-1}{r}$$

thus defining the coefficients $L(r, k)$. These numbers satisfy the difference equation

$$L(m, n)=(m-n+1)L(m-1, n-1)+nL(m-1, n)$$

with the initial conditions $L(m, 1)=1$, $m\geqq 1$, $L(1, n)=0$, $n\geqq 2$. The connection between $L(r, k)$ and the prepared Bernoulli numbers [see the following review] is given by

$$\sum_{k=1}^{2m+1}(-1)^{k+1}L(2m+1, k)=(-1)^m S_{2m+1}.$$

For Euler numbers S_{2m} the author gives

$$\sum_{k=1}^{2m+1}(-1)^{k+1}L'(2m+1, k)=(-1)^m S_{2m}$$

where

$$L'(r, k)=\sum_{n=1}^{r-k+1}\binom{r-1}{n-1}L(r-n, k-1).$$

If

$$L''(r, k)=\sum_{n=1}^{r-k+1}\binom{r}{n}L(r-n, k-1),$$

then

$$\sum_{k=1}^{m}(-1)^k L''(m, k)=(\sin \tfrac{1}{2}\pi m+\cos \tfrac{1}{2}\pi m)S_m.$$

An inversion of (1) gives

$$L(r, k)=\sum_{n=0}^{k-1}(-1)^n\binom{r+1}{n}(k-n)^r$$

which is a special case of the function

$$L(r, k, s)=\sum_{n=0}^{k-1}(-1)^n\binom{r+s}{n}(k-n)^r$$

whose properties are explored. Small tables of all four functions are given. *D. H. Lehmer* (Los Angeles, Calif.).

Singh, Daljit. On the divisibility of Eulerian and prepared Bernoullian numbers by prime numbers. Math. Student 20, 71–73 (1952).

The author considers the positive integers S_n generated by

$$\sec x+\tan x=\sum_{n=0}^{\infty}S_n x^n/n!,$$

so that S_{2n} are the non-zero Euler numbers $S_{2n}=|E_{2n}|$, while $S_{2n-1}=(2^{2n}-1)4^n|B_{2n}|/(2n)$ are the "prepared" Bernoulli numbers. A proof is given of the congruence

$$S_{p+r-1}\equiv(-1)^{(p-1)/2}S_r \pmod p \quad (r=1, 2, \cdots, p-2)$$

where p is an odd prime. This result is due to Kummer and Lucas in the respective cases r even and r odd. The congruence

$$(-1)^{(p+1)/2}S_{p-2}\equiv\sum_{r=1}^{(p-1)/2}(2r-1)^{p-2} \pmod p$$

is derived and transformed from the results of a "previous paper" (no reference is given). This result is a special case of a theorem of Glaisher [E. Lehmer, Ann. of Math. (2) 39, 350–360 (1938), p. 352, equation 10].

D. H. Lehmer (Los Angeles, Calif.).

Rodosskiĭ, K. A. On some estimates of the quantities $L(1, \chi)$. Doklady Akad. Nauk SSSR (N.S.) 86, 889–891 (1952). (Russian)

Let D be a positive integer, $4 \log\log D/\log D<\eta<1/10$. The author shows that there are only $O(D^{8\eta}\log^8 D)$ L-series formed by characters mod D, which do not satisfy the inequality

$$|\log|L(1, \chi)||<O(1)+\log \eta^{-1}.$$

For the proof the author refers to a previous paper [Ukrain. Mat. Žurnal 3, 399–403 (1951)] (not accessible to the reviewer). The author also gives the following inequalities

$$L(1, \chi_D^{(2)})L(1, \chi_{dD}^{(2)})=O(\log d \log (dD)),$$
$$L(1, \chi_d^{(4)}\chi_D^{(2)})=O(\log^{\frac12} d \log^{\frac12} (dD)),$$
$$1=O(\log D)(L(1, \chi_D^{(2)})+L(1, \chi_{dD}^{(2)})),$$

where $\chi^{(2)}$ and $\chi^{(4)}$ denote quadratic and quartic characters and where in the last inequality

$$(d, D)=1, \quad \log d<\log D/\log\log D.$$

H. Heilbronn (Bristol).

Specht, Wilhelm. Zur Zahlentheorie der Polynome. IV. Math. Z. 57, 291–335 (1953).

[For parts I–III see these Rev. 14, 251.] Let

$$f(z, a)=\sum_{(\mu)}a_\mu z_0^{\mu_0}z_1^{\mu_1}\cdots z_k^{\mu_k}, \quad \mu_0+\mu_1+\cdots+\mu_k=m$$

denote a polynomial of degree m with rational integral coefficients with the g.c.d. of the coefficients $=1$. Let $K^*_{m,k}(x)$ denote the number of such polynomials satisfying $0<N(f)\leqq x$, where $N(f)=(\sum_{(\mu)}a_\mu^2)^{1/2}$; also let $I^*_{m,k}(x)$ denote the number of irreducible polynomials and $R^*_{m,k}(x)$ the number of reducible polynomials. In previous papers the writer has discussed the distribution of normalized polynomials in a single indeterminate. Of the four principal results of the present paper we quote the following two.

Theorem 1. For $k\geqq 2$, $m\geqq 2$, $n=\binom{m+k}{m}$

$$K^*_{m,k}(x)=\frac{\alpha_n}{2\zeta(n)}x^n+O(x^{n-2}),$$

$$I^*_{m,k}(x)=\frac{\alpha_n}{2\zeta(n)}x^n+O(x^{n-2}),$$

$$R^*_{m,k}(x)=O(x^{n-2}),$$

where $\alpha_n=\pi^{n/2}/\Gamma(n/2+1)$. It is noted that for $n\geqq 5$ the error terms cannot be improved.

Theorem 2. For $k=1$, $m\geqq 5$,

$$K^*_{m,1}(x)=\frac{\alpha_{m+1}}{2\zeta(m+1)}x^{m+1}+O(x^{m-1}),$$

$$R^*_{m,1}(x)=\frac{\alpha_m\eta(m)}{4\zeta^2(m)}x^m+O(x^{m-1}),$$

$$I^*_{m,1}(x)=\frac{\alpha_{m+1}}{2\zeta(m+1)}x^{m+1}-\frac{\alpha_m\eta(m)}{4\zeta^2(m)}x^m+O(x^{m-1}),$$

where $\eta(m)$ is a certain constant. *L. Carlitz.*

Kanold, Hans-Joachim. Abschätzungen bei Kreisteilungspolynomen und daraus hergeleitete Bedingungen für die kleinsten Primzahlen gewisser arithmetischer Folgen. Math. Z. 55, 284–287 (1952).

The author proves that the least prime q of the arithmetic progression $mx+1$ with $m\geqq 4$ is less than or equal to $5\cdot 2^{\varphi(m)-3}$ where $\varphi(m)$ denotes Euler's function. This result is improved further in the cases m odd, $m\equiv 2 \pmod 4$, and $m\equiv 0 \pmod 4$, respectively. *A. Brauer.*

Dénes, Peter. Proof of a conjecture of Kummer. Publ. Math. Debrecen 2, 206–214 (1952).

Kummer [Math. Abh. Königl. Akad. Wiss. Berlin 1857, 41–74 (1858)] stated without proof the following theorem. If n is a non-negative rational integer, p an odd prime, $\zeta=\exp(2\pi i/p)$, $\Omega(\zeta)$ the field of pth roots of unity over the rational number field, ϕ and ϕ_1 integers prime to p in $\Omega(\zeta)$ satisfying the congruence $\phi\equiv\phi_1 \pmod{p^{n+1}}$, and if k is a

rational integer with $(p-1)\nmid k$, then

$$D_{kp^s}\log\phi(e^v)\equiv D_{kp^s}\log\phi_1(e^v)\pmod{p^{s+1}},$$

where the symbol $D_m\log\phi(e^v)$ denotes the value of the mth derivative of $\log\phi(e^v)$ with respect to v at $v=0$, and $\phi(e^v)$ results by setting e^v instead of ζ in $\phi(\zeta)$. In this paper the author gives the first complete proof of Kummer's theorem. Special cases of the theorem have previously been obtained by Vandiver [Bull. Amer. Math. Soc. **28**, 400–407 (1922); Trans. Amer. Math. Soc. **32**, 391–408 (1930)] and Uspensky [unpublished]. The proof is based upon a detailed and penetrating analysis of the congruence properties of the kth derivative of $\log F(e^v)$, where $F(e^v)$ is a polynomial in e^v with rational integral coefficients. There are seven lemmas too long to quote. The arguments are elementary in nature, and ideas of algebraic number theory are used only in the concluding phases of the proof. *A. L. Whiteman.*

Terada, Fumiyuki. On the principal genus theorem concerning the abelian extensions. Tôhoku Math. J. (2) **4**, 141–152 (1952).

This paper is concerned with a generalization to the Abelian case of the principal genus theorem for ideals in fields of algebraic numbers, as formulated and proved by Hasse for the case of a cyclic extension. This theorem may be generalized to arbitrary normal extensions in the following form: the first cohomology group of the idèle-class group is trivial. However, the results of the present paper yield more (in the Abelian case) than this general statement. Let k be an algebraic number field, and let K/k be a finite Abelian extension of k. Let $\mathfrak{f}$ be the conductor of K/k, and $\mathfrak{F}$ its divisor of genera: if $\mathfrak{P}$ is a prime ideal of K, the contribution of $\mathfrak{P}$ to $\mathfrak{F}$ is $\mathfrak{P}^v$, where v is the largest integer such that there exists in the Galois group G of K/k an operation $s\neq 1$ such that $s\cdot x\equiv x\pmod{\mathfrak{P}^v}$ for all integers x of K. Let $\mathfrak{m}$ be any integral divisor of k, and R the "ray" of k modulo $\mathfrak{m}\mathfrak{f}$, i.e., the group of principal ideals which may be represented by numbers $a\equiv 1\pmod{\mathfrak{m}\mathfrak{f}}$ of k. Let $s\to\mathfrak{A}(s)$ be a 1-cochain of G in the group of ideals of K whose coboundary takes its values in R. Then the author proves that $\mathfrak{A}(s)=\mathfrak{B}^{1-s}(A(s))$, where $\mathfrak{B}$ is an ideal and $s\to A(s)$ a 1-cochain with values in K^* whose coboundary takes its values in the group S of numbers $\equiv 1\pmod{\mathfrak{m}\mathfrak{f}}$ of k and is such that $A(s)\equiv 1\pmod{\mathfrak{m}\mathfrak{F}}$ for every $s\in G$. In general, if $s\to A(s)$ is any cochain with values in K^* whose coboundary takes its values in S and which satisfies the condition $(A(s))^{1-t}=(A(t))^{1-s}$ for any s, t in G, then the author proves that $A(s)=B^{1-s}A'(s)$, with $B\in K^*$, $A'(s)\equiv 1\pmod{\mathfrak{m}\mathfrak{F}}$. An essential feature of the proof is the use of a type of normalized cochains for an abelian group G, relative to a decomposition of G into a product of cyclic groups. The technical details are rather complicated, but could be greatly simplified by making use of the notion of idèle and of the form indicated above of the principal genus theorem. The following misprints may be troublesome to the reader: Lemma 1, 2d formula, read $\mathfrak{F}(K'/k)$ instead of $\mathfrak{F}(K'/K)$; lemma 2, part 1, read $A\equiv 1\pmod{\mathfrak{m}\mathfrak{F}}$ instead of $A\equiv 1\pmod{\mathfrak{m}\mathfrak{f}}$; lemma 4, read (m) instead of $(\mathfrak{m})$.

C. Chevalley (New York, N. Y.).

Jehne, Wolfram. Idelklassenfaktorensysteme und verallgemeinerte Theorie der verschränkten Produkte. Abh. Math. Sem. Univ. Hamburg **18**, 70–98 (1952).

Let Ω be an algebraic number field and K its Galois extension. A system $\bar{A}^*=\{A_\mathfrak{p}^*\}$ of local algebras $A_\mathfrak{p}^*$ over $\Omega_\mathfrak{p}$ is called an imbedding algebra-idèle for a central simple algebra A over K, when $A_\mathfrak{p}=A\times_\Omega\Omega_\mathfrak{p}$ can, for each $\mathfrak{p}$, be imbedded in $A_\mathfrak{p}^*$ so that $A_\mathfrak{p}$ is the commutator of $K_\mathfrak{p}$ in $A_\mathfrak{p}^*$. In such A we obtain Teichmüller's [Deutsche Math. **5**, 138–149 (1940); these Rev. **2**, 122] semi-factor-set Θ (on the Galois group of K/Ω) (with Teichmüller's 3-cocycle as its (semi-) coboundary). Then there are, for each $\mathfrak{p}$, a splitting semi-factor-set $\Theta^{(\mathfrak{p})}$ in $A_\mathfrak{p}^*$ and a 2-cochain $a^{(\mathfrak{p})}$ in $K_\mathfrak{p}$ such that $\Theta=\Theta^{(\mathfrak{p})}a^{(\mathfrak{p})}$. $\{a^{(\mathfrak{p})}\}$ gives rise to an idèle-class factor-set, in K/Ω, whose class is determined uniquely by the (locally defined) class of $\bar{A}^*$. Among the idèle-class 2-cohomology classes for K/Ω obtained in this manner, there is a canonical one, derived from $\bar{A}^*$ with $\sum(A_\mathfrak{p}^*/\mathfrak{p})=1/(K:\Omega)$, which is a generator of the group of such (as a matter of fact, all [cf. Hochschild and Nakayama, Ann. of Math. (2) **55**, 348–366 (1952); these Rev. **13**, 916]) idèle-class 2-cohomology classes and which is indeed (inverse to, in some formulations) the one derived by Weil [J. Math. Soc. Japan **3**, 1–35 (1951); these Rev. **13**, 439] and the reviewer [Ann. of Math. **55**, 73–84 (1952); these Rev. **13**, 629] by different methods. This canonical 2-cohomology class is used then, partly paralleling Nakayama [loc. cit.], to derive the Artin-Chevalley reciprocity. It is observed that the present construction uses, arithmetically, only Hasse's sum formula (while the reviewer's used the cyclotomic cyclic case of the reciprocity, too). Another advantage is that it gives at the same time the relationship to the Teichmüller-MacLane [MacLane, Bull. Amer. Math. Soc. **54**, 328–333 (1948); these Rev. **10**, 5] group and class [cf. Hochschild-Nakayama, loc. cit., and Nakayama, Ann. of Math. **57**, 1–14 (1953); these Rev. **14**, 453; the latter is particularly closely related to the present paper] (although the cohomology theory seems to be more effective in certain other respects; cf. Tate, ibid. **56**, 294–297 (1952); these Rev. **14**, 252). Finally, Hasse's [Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1947**, no. 8; these Rev. **11**, 155] Wiederspiegelungsproblem is settled by the use of the canonical class.

T. Nakayama (Nagoya).

Barnes, E. S. Isolated minima of the product of n linear forms. Proc. Cambridge Philos. Soc. **49**, 59–62 (1953).

Let $X_i=\sum_{j=1}^n a_{ij}u_j$ $(i=1,\cdots,n)$ with real a_{ij} and $\Delta=\det(a_{ij})\neq 0$, and let $M(X)$ be the lower bound of $|X_1\cdots X_n|$ over all integer sets $(u)\neq(0)$. Define γ_n as the upper bound of $M(X)/|\Delta|$ over all sets (X_i). It is known [see Davenport and Rogers, Philos. Trans. Roy. Soc. London. Ser. A. **242**, 311–344 (1950); these Rev. **12**, 394] that γ_n is finite, and that (for $n=2$ or 3) $M(X)=\gamma_n|\Delta|$ holds only when the set (X_i) is unimodularly equivalent to a multiple of a certain critical set (W_i) given by $W_i=\sum_{j=1}^n\omega_j^{(i)}u_j$, where $\omega_1,\cdots,\omega_n$ is an integral basis for the totally real algebraic number field of degree n and least discriminant, and where superscripts denote conjugates. Further, there exists a constant $\gamma_n^{(2)}<\gamma_n$ such that $M(X)\leqq\gamma_n^{(2)}|\Delta|$ for sets (X_i) not satisfying the above condition.

The author considers sets of forms analogous to the above critical set, and generalizes results of Davenport and Rogers [loc. cit.] to obtain the following theorem: Let $X_i=\sum_{j=1}^n\alpha_j^{(i)}u_j$ $(i=1,\cdots,n)$, where $\alpha_1,\cdots,\alpha_j$ are elements of a totally real algebraic field of degree n, and suppose that there exist integers (u) with $|X_1\cdots X_n|=M(X)$ for which $X_1,\cdots,X_n$ have arbitrarily prescribed signs. An ϵ-neighborhood of the set (X_i) is gotten by varying each coefficient $\alpha_j^{(i)}$ by an arbitrary real number β_{ij} subject to $|\beta_{ij}|<\epsilon$. Then there exists a number $\delta>0$ such that for

each set (Y_i) in a sufficiently small ϵ-neighborhood of (X_i), either $(Y_i)=\text{const.}\times(X_i)$ or $M(Y_i)<(1-\delta)M(X_i)$.

I. Reiner (Urbana, Ill.).

Barnes, E. S., and Swinnerton-Dyer, H. P. F. The inhomogeneous minima of binary quadratic forms. I. Acta Math. **87**, 259–323 (1952).

Let $f(P)=f(x, y)=ax^2+bxy+cy^2$ be a real indefinite form, where P represents the real point (x, y). Write $(x_1, y_1)\equiv(x_2, y_2)$ if x_1-x_2 and y_1-y_2 are integers, and define $M(f, P)=\inf|f(Q)|$ over all $Q\equiv P$, and $M(f)=\sup M(f, P)$ over all P. $M(f)$ is called the inhomogeneous minimum of f; it is the inf of those numbers m such that for any P_0 there exists $P\equiv P_0$ with $|f(P)|\leqq m$. If $M(f)$ is itself such an m, it is said to be an attained minimum, and a theorem of Heinhold [Math. Z. **44**, 659–688 (1939), p. 660] guarantees the existence of a point P with $M(f, P)=M(f)$, and the further existence of a corresponding point $Q\equiv P$ with $|f(Q)|=M(f)$. The authors also consider $M_2(f)=\sup M(f, P)$ taken over all P for which $M(f, P)<M(f)$. If $M_2(f)<M(f)$, then $M(f)$ is called an isolated minimum, and $M_2(f)$ the second minimum. A criterion for the existence of isolated minima is given.

The evaluation of $M(f)$ is of significance in determining the existence of a Euclidean algorithm in the quadratic field $k(\sqrt{m})$, $m>0$; if $f_m(x, y)=\|x+\omega y\|$ is the norm of the general integer $x+\omega y$ of $k(\sqrt{m})$, then $k(\sqrt{m})$ is Euclidean if and only if $M(f_m, P)<1$ for all rational points P. Consequently, $k(\sqrt{m})$ is Euclidean if $M(f_m)<1$.

The authors evaluate $M(f_m)$ and $M_2(f_m)$ for many values of m, and tabulate these and the previously known results for $m\leqq 101$. Rédei's statement [Math. Ann. **118**, 588–608 (1942), p. 601, footnote; these Rev. **6**, 38] that for every rational point P either $M(f_{61}, P)=41/39$ or $M(f_{61}, P)<1$, is disproved, and it is shown that indeed

$$41/39=M_2(f_{61})<M(f_{61}).$$

The authors also prove that $k(\sqrt{97})$ is not Euclidean; the contrary was stated by Rédei [ibid., p. 607]. An example is given of a form with unattained first minimum.

Let $\phi(\alpha)$ be the inf of those numbers $m(\alpha)$ with the following property: for every real λ there exists an integer x such that $|(x+\lambda)^2-\alpha|\leqq m(\alpha)$. The authors evaluate $\phi(\alpha)$ explicitly, and show (trivially) that $M(f, P)\leqq|a|\cdot\phi(Dy^2/4a^2)$, where $P=(x, y)$. If K is taken as the supposed value of $M(f)$, the above considerations enable them to find a set R^* of points P where $M(f, P)<K$. Those exceptional points P where $M(f, P)\geqq K$ cannot transform (mod 1) into any point of R^* under any automorph of f. In some cases this permits the exceptional points to be determined easily; in the other cases the methods used are too complicated to describe here. The problem is then resolved into the evaluation of $M(f, P)$ at the exceptional points P. This is accomplished (in the simpler cases) by a lemma stating that if $M(f, P_i)<K$ $(i=1, \cdots, N)$, where $P_1, \cdots, P_N$ are N incongruent points which are permuted (mod 1) by some automorph of f, then there exists a point $P(x, y)\equiv P_i$ for some i, with $|f(P)|<K$, and for which $|y|$ cannot exceed an explicitly given bound depending on f, T and K. An extensive bibliography is given at the end of the paper.

I. Reiner (Urbana, Ill.).

★Inkeri, K. Non-homogeneous binary quadratic forms. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 216–224. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

This paper gives a survey of results connected with Minkowski's theorem on the product of two non-homogeneous linear forms. In particular, attention is drawn to the form $f(x, y)=x^2+xy-3y^2$, which has the property that the lower bound of $|f(x+x_0, y+y_0)|$ for integral x, y is always $\leqq\frac{1}{3}$, although for certain x_0, y_0 the inequality

$$|f(x+x_0, y+y_0)|\leqq\tfrac{1}{4}$$

is insoluble. [For later results and a list of references, see the paper reviewed above.] *H. Davenport* (London).

Brandt, H. Über Stammfaktoren bei ternären quadratischen Formen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **100**, no. 1, 24 pp. (1952).

Let $f=a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_2x_3+a_5x_1x_3+a_6x_1x_2$ be a ternary form with rational, or rational integral, coefficients a_i. The author points out that the arithmetical theory of such forms has been in an unsatisfactory state, requiring a distinction between properly and improperly primitive forms when the a_i are integral, and the consideration of numerous special cases, due to the nearly exclusive attention given to forms with even a_4, a_5, a_6, following Gauss. He avoids this restriction and suitably defines discriminant, coefficient-matrix, divisor of a form, adjoint form, primitive form, first and second invariants, etc., without it. [See his papers: (1) Math. Nachr. **6**, 315–318 (1952); (2) Math. Ann. **124**, 334–342 (1952); these Rev. **14**, 453, 454.] In particular, he defines the stem-factors (Stammfaktoren) of f as those primes common to the discriminants of all integral forms obtained from f by applying a non-singular rational transformation and multiplying by a rational factor. He proves that the stem-factors are the primes p for which, if f is integral, $f(x_1, x_2, x_3)\equiv 0 \pmod{p^n}$ has non-trivial solutions only for a finite number of exponents n. He has previously used them in studying the characters of f, obtaining much simpler results than those of Smith [see (2), above]. This paper is devoted chiefly to their determination, by means of values of Legendre symbols for odd primes, and by various criteria for the prime 2, with different cases according to the power of 2 dividing the invariants or their residues modulo 16. *R. Hull* (Lafayette, Ind.).

Lochs, Gustav. Über die Anzahl der Gitterpunkte in einem Tetraeder. Monatsh. Math. **56**, 233–239 (1952).

Denote by $n(x)=n(x; \alpha_1, \alpha_2, \cdots, \alpha_k)$ the number of lattice points inside or on the boundary of the k-dimensional "tetrahedron" bounded by the k coordinate hyperplanes $x_1=0, x_2=0, \cdots, x_k=0$ and the hyperplane

$$\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_kx_k=x,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_k$, and x are positive. The reviewer [Duke Math. J. **7**, 341–353 (1940); these Rev. **2**, 149] and the author [Jber. Deutsch. Math. Verein. **54**, 41–51 (1950); these Rev. **12**, 82] have independently considered the problem of obtaining two polynomials, P, Q, of degree k in x, with coefficients depending on the α's, for which $P(x)<n(x)<Q(x)$.

In the present paper the author studies two inequalities, the first of which is

$$n(x)<A^{-1}\left[\left(x+\frac{1}{2}\sum_{i=1}^{k}\alpha_i\right)^k-\frac{k(k-1)}{24}x^{k-2}\left(\sum_{i=1}^{k}\alpha_i^2\right)\right]-\Delta,$$

where $A = k!a_1a_2\cdots a_k$ and

$$\Delta = \sum B_1\{a_1^{-1}(x - g_2a_2 - \cdots - g_ka_k)\},$$

in which $B_1\{u\} = u - [u] - \frac{1}{2}$ is the first Bernoulli function and the sum extends over all sets of non-negative integers, for which the argument of B_1 is non-negative, that is, over the lattice points of a $(k-1)$-dimensional tetrahedron.

The ratio a_2/a_1 is considered as expanded in a continued fraction and inequalities for Δ are obtained in terms of the elements of the continued fraction and $n(x; a_2, \cdots, a_k)$. No very definitive results are obtained. *D. H. Lehmer.*

ANALYSIS

Moriguti, Sigeiti. A modification of Schwarz's inequality with applications to distributions. Ann. Math. Statistics 24, 107–113 (1953).

Given a function $\Phi(t)$, of bounded variation on a closed interval $[a, b]$ and continuous at both ends, let $\bar{\Phi}(t) = \sup F(t)$ for functions $F(t)$ that are convex and dominated by $\Phi(t)$ on $[a, b]$: the author calls $\bar{\Phi}(t)$ the greatest convex minorant of $\Phi(t)$ on $[a, b]$. Theorem 1: The relation

$$(1)\qquad \int_a^b x(t)d\Phi(t) \leqq \int_a^b x(t)\bar{\phi}(t)dt$$

holds for any (broadly) increasing function $x(t)$ for which the integrals exist and are finite, where $\bar{\phi}(t)$ is the right-hand derivative of $\bar{\Phi}(t)$, and there is equality in (1) if and only if $x(t)$ is constant in every interval where $\bar{\Phi}(t) < \min\{\Phi(t-0), \Phi(t+0)\}$ and $x(t) = x(t\pm 0)$ when

$$\Phi(t\mp 0) < \Phi(t\pm 0).$$

By combining (1) with Schwarz' inequality the author obtains his modification of the latter, namely

$$(2)\qquad \int_a^b x(t)d\Phi(t) \leqq \left(\int_a^b \{x(t)\}^2dt\right)^{\frac{1}{2}}\left(\int_a^b \{\bar{\phi}(t)\}^2dt\right)^{\frac{1}{2}}.$$

He notes that Čebyšev's inequality

$$\int x(t)y(t)dt\cdot\int dt \geqq \int x(t)dt\cdot\int y(t)dt,$$

for functions $x(t)$, $y(t)$ monotonic in the same sense on $[a, b]$, is obtainable from a particular case of (1). Next, let $F(x)$ be a cumulative distribution function, $x(F)$ be its inverse, and μ and σ be the corresponding mean and standard deviation. Various inequalities, some known and some new, are obtained easily as applications of (2). In particular, (i) if α and β are in $[0, 1]$, then

$$x(1-\beta) - x(\alpha) \leqq \sigma\sqrt{(\alpha^{-1}+\beta^{-1})},$$

which is related to the Bienaymé-Čebyšev inequality $1 - F(x) - F(-x) \leqq \sigma^2/x^2$; (ii) a method is given for computing upper bounds for $\mathcal{E}[X_i - \mu]$ and $\mathcal{E}[X_i - X_j]$, where X_i denotes the ith smallest member in a sample of size n, and for $n = 2m+1$, $m = 1, 2, \cdots, 9$, upper bounds so obtained for $\mathcal{E}[X_{m+1} - \mu]/\sigma$ are tabulated and found to be roughly half those yielded by the ordinary Schwarz inequality. [Remark. It is clear from the proof of (1) that $\bar{\Phi}(t)$ may be replaced by other (mildly restricted) minorants of $\Phi(t)$: for necessary and sufficient conditions on these (when $\Phi(t)$ is absolutely continuous) cf. Hardy, Littlewood, and Pólya, Inequalities, 1st ed., Cambridge, 1934, Theorem 399, p. 298.]

H. P. Mulholland (Birmingham).

Isaacs, G. L. M. Riesz's mean value theorem for infinite integrals. J. London Math. Soc. 28, 171–176 (1953).

Soit $f(t) \in L(x_0, X)$ pour tout $X \geqq x_0$ et $0 < \alpha < 1$. On suppose que l'intégrale $f_\alpha(y, x_0) = (1/\Gamma(\alpha))\int_{x_0}^{+\infty}(t-y)^{\alpha-1}f(t)dt$ converge $(y \leqq x_0)$. Alors pour $y < x_0$,

$$|f_\alpha(y, x_0)| \leqq \text{vrai}\max_{x_0\leqq u\leqq\infty}|f_\alpha(u, u)|.$$

Ce résultat généralise un théorème de M. Riesz [Acta Litt. Sci. Szeged 1, 114–126 (1923)]. *M. Zamansky* (Paris).

Berg, Lothar. Über eine Abschätzung von Mathieu. Math. Nachr. 7, 257–259 (1952).

The paper contains a proof of Mathieu's inequality

$$\sum_{n=1}^{\infty}\frac{n}{(n^2+c^2)^2} < \frac{1}{2c^2}$$

based upon Euler's summation formula. *J. Aczél.*

Laurenti, Fernando. Sopra alcune questioni di minimo. Atti Sem. Mat. Fis. Univ. Modena 5, 140–145 (1951).

The author shows that the minimal value of functions of the form

$$x^n + \sum_{k=1}^{s} a_kx^{n_k} \quad (a_k > 0)$$

can be calculated without making use of the calculus, by reduction to the theorem of the arithmetic and geometric means: If there exists a set of integers α_k such that

$$n + \sum_{k=1}^{s}\alpha_kn_k = 0$$

and

$$x^n = \frac{a_1}{\alpha_1}x^{n_1} = \frac{a_2}{\alpha_2}x^{n_2} = \cdots = \frac{a_k}{\alpha_k}x^{n_k}$$

although he applies his result also to problems where this condition is not fulfilled. The general result can be obtained, however, by simply proceeding to the limit after having made the obvious generalization to rational α's.

The author seems not to be acquainted with the works of P. P. Korovkin [Inequalities, Gostehizdat, Moscow-Leningrad, 1952, pp. 28–34; these Rev. 14, 24], who derives this result in the case $k = 1$ both if a is positive or negative, and of Ya. C. Perel'man [Zanimatel'naya algebra, 5th ed., Moscow-Leningrad, 1950, pp. 142–144], who applies a method quite similar to that used by the author.

J. Aczél (Debrecen).

Bellman, Richard. Notes on matrix theory. II. Amer. Math. Monthly 60, 173–175 (1953).

If Q is a positive definite quadratic form, one easily proves by transforming Q to its principal axes that

$$(1)\qquad \int_{-\infty}^{\infty}\exp(-Q)\prod dx_i = \pi^{n/2}|A|^{-n/2},$$

where $|A|$ is the determinant of the matrix A of Q. Replacing A by $\alpha A + (1-\alpha)B$ for A and B positive definite and applying Hölder's integral inequality yields

$$|\alpha A + (1-\alpha)B| \geqq |A|^\alpha|B|^{1-\alpha}.$$

Making suitable changes in sign in the x_i in (1), adding the results, and using the arithmetic-geometric mean inequality gives $\prod a_{ii} \geqq |A|$. *W. Givens* (Knoxville, Tenn.).

R.-Salinas, Baltasar. Note on a generalization of the formulas of Taylor, Darboux and Euler-Maclaurin. Revista Mat. Hisp.-Amer. (4) 12, 281–289 (1952). (Spanish)

Let T be the differential operator $D^{n+1}+\sum_0^n C_j(x)D^j$ and let L be the linear functional such that

$$L(\phi)=\sum_0^n[\beta_j\phi^{(j)}(b)-\alpha_j\phi^{(j)}(a)].$$

Suppose that there are $n+1$ independent functions $\phi_0, \cdots, \phi_n$, such that $T(\phi_i)=0$, $L(\phi_i)=0$. Then, there is a function ψ, formed from the ϕ_i, such that for every $f\in C^{n+1}$, $L(f)=\int_a^b T(f)\psi$. By special choices of the C_j or the ϕ_i, Taylor's formula and various generalizations of it may be obtained. A related approach to Taylor's formula may be found in a note by D. C. Lewis [Amer. Math. Monthly 59, 692–693 (1952)]. *R. C. Buck* (Madison, Wis.).

Picone, Mauro. Points de vue généraux sur l'interpolation et quelques recherches qu'ils suggèrent. Rend. Circ. Mat. Palermo (2) 1, 240–259 (1952).

Exposé des résultats obtenus par l'auteur dans son mémoire: Ann. Scuola Norm. Super. Pisa (3) 5, 193–244 (1951) [ces Rev. 14, 144]. Quelques problèmes nouveaux sont posés et quelques applications nouvelles aux fonctions holomorphes sont données. *J. Favard* (Grenoble).

Whitney, A. M. A reduction theorem for totally positive matrices. J. Analyse Math. 2, 88–92 (1952). (Hebrew summary)

A matrix $A=\|a_{ij}\|$ is called k-times (strictly) positive if all its minors of order ν, $1\leqq\nu\leqq k$, are non-negative (positive). An $m\times n$ matrix is (strictly) totally positive if it is k-times (strictly) positive for $k=\min(m, n)$. The author shows that to a given totally positive matrix and a given $\epsilon>0$ there exists a strictly totally positive matrix B such that $0\leqq b_{ij}-a_{ij}<\epsilon$ for all i, j. This theorem is used to derive a reduction theorem which allows one to decide if a given $m\times n$ matrix is totally positive by making the same decision for an appropriate $m\times(n-1)$ matrix. *E. Hille.*

Aissen, Michael, Schoenberg, I. J., and Whitney, A. M. On the generating functions of totally positive sequences. I. J. Analyse Math. 2, 93–103 (1952). (Hebrew summary)

A sequence $\{a_n\}$, $a_0=1$, $a_n=0$ when $n<0$, is said to be totally positive if the two-way infinite matrix $A=\|a_{i-j}\|$ is totally positive. Let $f(z)=\sum_0^\infty a_nz^n$ be the generating function of $\{a_n\}$. Starting from the observation that the function

$$F(z)=e^{\gamma z}\prod_{\nu=1}^\infty(1+\alpha_\nu z)\prod_{\nu=1}^\infty(1-\beta_\nu z)^{-1},$$

$\gamma\geqq 0$, $\alpha_\nu\geqq 0$, $\beta_\nu\geqq 0$, $\sum(\alpha_\nu+\beta_\nu)<\infty$, generates a totally positive sequence, the authors try to prove the converse theorem. They succeed in showing that the infinite product has the desired form, but instead of the factor $e^{\gamma z}$ they get merely $e^{g(z)}$ where $g(z)$ is an entire function, $g(0)=0$, and the exponential also generates a totally positive sequence. A number of devices are used in the reduction such as earlier results of I. J. Schoenberg [Courant Anniversary Volume, Interscience, New York, 1948, pp. 351–370, these Rev. 9, 337] concerning totally positive sequences, the fact that if $B=\|b_{ij}\|$ is totally positive so is $C=\|c_{ij}\|$ where $B^{-1}=\|(-1)^{i+j}c_{ij}\|$, and the reduction theorem of A. M. Whitney in the preceding review. *E. Hille.*

Edrei, Albert. On the generating functions of totally positive sequences. II. J. Analyse Math. 2, 104–109 (1952). (Hebrew summary)

[Cf. the preceding review.] The author gives two proofs of the fact that $g(z)=\gamma z$, thus completing the proof of the converse theorem of Aissen, Schoenberg, and Whitney. The second proof is based on the following theorem: If $g_1(z)$ is an entire odd function, if $e^{g_1(z)}=\sum b_nz^n$, and if $\det\|b_{n-i-j}\|_{1,n}>0$ for each n, then $g_1(z)=c_1z$, $c_1>0$. The proof of this follows from the theory of continued fractions and Grommer's investigations of the moment problem. *E. Hille* (New Haven, Conn.).

Edrei, Albert. Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence. Canadian J. Math. 5, 86–94 (1953).

M. Aissen, I. J. Schoenberg, and A. Whitney [see the second preceding review] have proved the following theorem: Let $a_0, a_1, \cdots (a_0>0)$ be such that the matrix $A=\|a_{i-j}\|$ $(i, j=1, 2, \cdots; a_n=0$ if $n<0)$ is totally positive, i.e., all its minors of all orders are non-negative. Then the generating function $f(z)=\sum a_nz^n$ is of the form

$$f(z)=\exp\{h(z)\}\prod_1^\infty(1+\alpha_nz)/\prod_1^\infty(1-\beta_nz), \tag{1}$$

where $\alpha_n\geqq 0$, $\beta_n\geqq 0$, $\sum(\alpha_n+\beta_n)<\infty$, while $h(z)$ is entire and such that $\exp\{h(z)\}$, if expanded in powers of z, will also generate a totally positive sequence. Edrei [see the preceding review] then proved the decisive result that $h(z)=\gamma z+\delta$ $(\gamma\geqq 0)$. The present paper which is self-contained presents an earlier version of the author's work which is independent of the previous work of Aissen, Schoenberg, and Whitney. Outline of procedure: It is no essential restriction to assume that the minors $A_{m,n}=\det\|a_{i-j}\|$ $(i=m+1, \cdots, m+n, j=1, \cdots, n)$ are all positive for $m\geqq 0$, $n\geqq 1$. With this assumption the denominators of Padé's table associated with $f(z)$ exhibit remarkable limit properties which eventually lead to the representation (1). As in Edrei's other proof, the application of a theorem of R. Nevanlinna completes the proof. *I. J. Schoenberg* (Swarthmore, Pa.).

Schoenberg, I. J., and Whitney, Anne. On Pólya frequency functions. III. The positivity of translation determinants with an application to the interpolation problem by spline curves. Trans. Amer. Math. Soc. 74, 246–259 (1953).

[For Parts I and II in this series, see Schoenberg, these Rev. 13, 923; 12, 23]. A non-negative function $\Lambda(x)$ in $L(-\infty, \infty)$ is called a Pólya frequency function if for any two sets of increasing numbers

$$x_1<x_2<\cdots<x_n,\quad y_1<y_2<\cdots<y_n,\quad n=1, 2, \cdots,$$

we have the inequality $D\equiv\det\|\Lambda(x_i-y_j)\|_{1,n}\geqq 0$. It was shown in Part I that the bilateral Laplace transform of $\Lambda(x)$ is the reciprocal of an entire function $\Psi(s)$ of the form

$$\Psi(s)=Ce^{-\gamma s^2+\delta s}\prod_{\nu=1}^\infty(1+\delta_\nu s)e^{-\delta_\nu s},$$

where $C>0$, $\gamma\geqq 0$, δ, δ_ν are real, $0<\gamma+\sum\delta_\nu^2<\infty$. Case 1 is that in which $\Psi(s)=\prod(1+\delta_\nu s)$, $\sum|\delta_\nu|<\infty$, case 2 that in which $\gamma>0$ or $\gamma=0$ and $\sum|\delta_\nu|=\infty$. It is shown in the present paper that D is always positive in case 2. In case 1 let k be the number of $\delta_\nu>0$, h the number of $\delta_\nu<0$. Then $D>0$ if and only of $x_{\nu-k}<y_\nu<x_{\nu+h}$ for $\nu=1, \cdots, n$ with the convention that $x_r=-\infty$ if $-\infty\leqq r<1$ and $x_r=+\infty$ if

$n<r\leqq+\infty$. The proof involves the consideration of a large number of subcases. If $x_+=\max(0, x)$, $\Lambda(x)=e^{-x}(x_+)^k$ is a Pólya frequency function belonging to case 1 and the main theorem shows that $\det \|[(x_i-y_j)^k]>0$ if and only if $x_{\nu-k-1}<y_\nu<x_\nu$, $\nu=1, \cdots, n$. A spline function of degree k is an expression of the form $F(x)=P_k(x)+\sum_{\nu=1}^n A_\nu[(x-\xi_\nu)_+]^k$ where $P_k(x)$ is an arbitrary polynomial of degree $\leqq k$ and the ξ_ν are the n given knots. The interpolation problem consists in deciding if a spline curve can be laid through $n+k+1$ given points (x_ν, Y_ν), the degree k and the knots ξ_ν being given. Necessary and sufficient conditions are given with the aid of the special result mentioned above.

E. Hille (New Haven, Conn.).

Theory of Sets, Theory of Functions of Real Variables

Popadić, Milan S. A characteristic property of finite sets. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 4, no. 6, 8 pp. (1951). (Serbo-Croatian. English summary)

A set is finite if and only if it can be expressed as a simply ordered set M such that, for any set N, the relation $M\subseteq N$ holds if (1) there exists a closed interval A of M with $O\subset A\subseteq M$ and $A\subseteq N$, and (2) for any closed interval B of M with $O\subset B\subset M$ and $B\subseteq N$, there exists a closed interval C of M with $B\subset C\subseteq M$ and $C\subseteq N$. *F. Bagemihl.*

Norris, Michael J. Cofinally concentrated directed systems. J. Computing Systems 1, 81–85 (1953).

The author calls a directed system A cofinally concentrated if for every element a of A it is true that the cardinal number of the set of elements b such that $a\leqq b$ is false is less than the cardinal number of A. His principal theorem states that if A is a cofinally concentrated directed system, and Ω is the first ordinal number associated with the cardinal number of A, then A has a well-ordered cofinal subset whose ordinal number $f(\Omega)$ is the first ordinal associated with a cofinal subset of the set of all ordinals less than Ω.

It follows that if the cardinal number of a subset of A which is both cofinal and cofinally concentrated is regular, it is the smallest cardinal number of cofinal subsets; that there is at most one regular cardinal number of subsets both cofinal and cofinally concentrated; and that if the cardinal number of a cofinally concentrated directed system is regular, there is no cofinal subset with smaller cardinal number. These statements are false, however, for irregular cardinal numbers. *O. Frink* (State College, Pa.).

Bloch, Gérard. Sur les ensembles stationnaires de nombres ordinaux et les suites distinguées de fonctions régressives. C. R. Acad. Sci. Paris 236, 265–268 (1953).

An announcement of results, some of which, as the author notes, have already been proved in more general form by Neumer [Math. Z. 54, 254–261 (1951); these Rev. 13, 331], concerning certain nonenumerable subsets, termed "stationary", of $W(\omega_1)$ having the property that every ordinal-valued function ϕ defined on such a subset and satisfying $\phi(\alpha)<\alpha$ for every α satisfies $|\phi^{-1}(\alpha)|=\aleph_1$ for some α, as well as applications to distinguished sequences $\{\phi_n(\xi)\}_{n<\omega}$ of ordinals associated with, e.g., the limit numbers ξ of $W(\omega_1)$, where, for each n, $\phi_n(\xi)$ is constant over certain stationary sets (which depend on n), $\aleph_1$ in number. *F. Bagemihl.*

Zakon, Elias. Left side distributive law of the multiplication of transfinite numbers. Riveon Lematematika 6, 28–32 (1953). (Hebrew. English summary)

The following theorem is established: given three ordinal numbers α, β, γ such that $\alpha+\beta\neq\beta\neq0\neq\gamma$ there is a unique non-negative integer c such that $(\alpha+\beta)\gamma=\alpha\gamma+\beta c$.

S. Eilenberg (New York, N. Y.).

Kondô, Motokiti, et Tugue, Tosiyuki. Quelques cribles par rapport aux mesures. J. Math. Tokyo 1, 55–61 (1951).

Nous envisageons des ensembles $E\subset$plan réel XOY. Pour tout ensemble E et pour tout point $x\in OX$, on pose: $E^{(x)}$=ensemble de tous les points $p\in E$ tels que p soit sur la parallèle à OY passant par x. Puis, pour toute propriété P qu'un ensemble linéaire peut ou non posséder, et pour tout ensemble E, on pose: $\Gamma_P(E)$=ensemble de tous les points $x\in OX$ tels que $E^{(x)}$ possède la propriété P. Dans la suite, P désignera la propriété: "être de mesure >0 au sens de Lebesgue". L'A., améliorant un résultat de D. Montgomery [Amer. J. Math. 56, 569–586 (1934)], démontre le théorème suivant. Théorème: Si E est un ensemble A (resp. CA), alors $\Gamma_P(E)$ est aussi un ensemble A (resp. CA). De plus, pour tout ensemble H qui est linéaire et A (resp. CA), il existe un ensemble E qui est A (resp. CA) et qui vérifie $H=\Gamma_P(E)$. L'A. donne d'intéressants corollaires et généralisations de ce théorème. *A. Appert* (Angers).

Newman, M. H. A. Path-length and linear measure. Proc. London Math. Soc. (3) 2, 455–468 (1952).

The paper concerns relations between the linear measure of a continuum X and the length of an embedded closed path f which the author terms a loop (to the dismay of algebraists and of the reviewer); f is on X if it fills X. The statement that X has linear measure $\leqq L<\infty$ is shown to be equivalent to the existence of such an f on X of length $\leqq 2L$, which has almost everywhere multiplicity $\geqq 2$, or alternately, if X is in Euclidean m-space, which is null-homotopic in X. The similar problem, due to Ważewski, for paths not restricted to be closed, was treated, for instance, by Eilenberg and Harrold [Amer. J. Math. 65, 137–146 (1943), Theorem 2, p. 139; these Rev. 4, 172], but in a less complete form. In the case in which X is a plane continuum of 2-dimensional measure zero, the author establishes further an inequality between the length of X and that of loops on the frontiers of its residual domains and also a converse theorem; this inequality closely resembles one due to the reviewer [Fund. Math. 35, 275–302 (1948), Theorem (7.2), p. 294; these Rev. 10, 520]. *L. C. Young.*

Davies, R. O. Subsets of finite measure in analytic sets. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 488–489 (1952).

A Souslin set of infinite Hausdorff Λ^s-measure contains closed subsets of all finite Λ^s-measures. This extends a result of Besicovitch [same Proc. 14, 339–344 (1952); these Rev. 14, 28]. *H. D. Ursell* (Leeds).

***Mayrhofer, Karl. Inhalt und Mass.** Springer-Verlag, Wien, 1952. viii+269 pp. Broschiert, \$8.60; Ganzleinen, \$9.30.

This book gives a very systematic, clear, and readable presentation of the theory of content and measure. In Chapter I the content i is defined as a non-negative, additive set-function on a field $\mathfrak{k}$ of sets, vanishing for the empty set O, and the measure m as a non-negative, completely

additive set-function on a σ-field $\mathfrak{k}$ with $m(O)=0$. The case of a completely additive content i (again defined on a field $\mathfrak{k}$) is also considered. The outer and inner contents of a set M are introduced by $\bar{i}(M)=\inf i(A)$ with $M\subset A \in \mathfrak{k}$ and $\underline{i}(M)=\sup i(A)$ with $M\supset A \in \mathfrak{k}$; analogously for the outer and inner measure $\bar{m}(M)$ and $\underline{m}(M)$. The extension of a content or measure to a complete content or measure and of a completely additive content to a measure (cf. also §23 in Chapter V) are discussed, as well as content and measure in product spaces (using the method of H. Hahn).

Chapter II applies the general theory of Chapter I to the n-dimensional Jordan content (also for unbounded sets) starting from the elementary content i of aggregates of cubes and then going to the smallest complete content j over i. Analogously in Chapter III, extending the same i to the smallest measure (which is incomplete) one obtains the Borel measure, while the extension of i to the smallest complete measure gives the n-dimensional Lebesgue measure. Then a detailed discussion of the Lebesgue measure is given including the Vitali covering theorem and the density of sets. Chapter IV treats the linear transformations of content and measure.

Chapter V discusses Carathéodory's measure theory starting with Carathéodory's outer measure (here defined on a σ-field) and finally leading to the reviewer's characterization of the inner measure. [In defining the outer measure (No. 91) the postulates A_2 and A_3 should be introduced immediately after A_1, certainly before theorem 1; otherwise $\mu(A)$ and $\mu^*(AM)$ could be infinite of opposite signs, and then (2) would be meaningless.] The relations to the theory of Chapter I are stressed by the author. [Cf. also Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **158**, 1–36 (1950); these Rev. **12**, 683.]

In Chapter VI Boolean algebras and σ-algebras are systematically discussed and their isomorphisms and homomorphisms are studied. The elements of a Boolean algebra are called somas, following C. Carathéodory. This chapter is essentially influenced by C. Carathéodory's papers on the theory of somas. Content and measure are defined as functions of somas. The greatest part of the Chapters I and V was developed in such a manner that the generalization to somas is immediately possible. Finally an appendix discusses the theory of Borel sets (which were occasionally used in previous chapters). *A. Rosenthal.*

***Munroe, M. E. Introduction to measure and integration.** Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1953. x+310 pp. $7.50.

This is a textbook designed "to present measure theory from the abstract or postulational point of view and yet to do this in such a way that the graduate student as well as the expert will find the work helpful".

Chapter I (Point set theory), which occupies nearly a quarter of the book, sets forth such preliminary matter, beyond the fundamentals of real function theory, as is needed; its topics include algebra of sets, metric spaces, function spaces, linear spaces, additive classes, and Borel sets. Chapter II (Measure—general theory) deals with completely additive set functions (in particular, measure functions), outer measures and measurability with respect to them, construction of outer measures, regular outer measures, and metric (or Carathéodory) outer measures. Chapter III (Measure—specific examples) deals with Lebesgue-Stieltjes measures and their application to probability, Hausdorff measures, Haar measure, and non-measurable sets. Chapter IV (Measurable functions) gives the basic properties of measurable functions and goes on to approximation theorems. A function that assumes only a finite number of (finite) values, each on a measurable set, is called a simple function, and it is proved that every non-negative measurable function is the limit of a (broadly) increasing sequence of non-negative simple functions. There follow Egoroff's and Lusin's Theorems and a short section on stochastic variables. Chapter V (Integration): the (Lebesgue) integral of a simple function is defined in the obvious way; a non-negative function is called integrable on a set E if it is the limit of a (broadly) increasing sequence of non-negative simple functions whose integrals on E are bounded; and a function of variable sign is treated as usual by splitting it into positive and negative parts. The chapter also treats the connexion between Riemann and Lebesgue integrals, elementary properties of the latter, absolute continuity, Hahn decomposition, the Radon-Nikodým theorem, Fubini's theorem, and the expectation of a stochastic variable. Chapter VI (Convergence theorems) treats uniform convergence, convergence almost everywhere, (strong) convergence in the mean of order p, the L_p spaces, linear functionals on Banach spaces, orthogonal expansions in Hilbert space, and the mean ergodic theorem. Chapter VII (Differentiation) treats Vitali's covering theorem, upper and lower derivatives, regular derivatives, strong derivatives, the relation between a Lebesgue-Stieltjes measure and the integral of its derivative, the Lebesgue decomposition theorem, metric density and approximate continuity, characterization of measurable functions as almost everywhere approximately continuous, and differentiation with respect to nets.

The treatment is modern without being forbiddingly "highbrow", and the author has been at great pains to smooth the reader's path, both by frequent informal explanations and by judicious concatenation of theorems and lemmas. The result is an easily readable textbook that is admirably adapted to its purpose. The reviewer's only criticism is on a minor point: the axiom of choice is stated in Chapter I and is invoked explicitly in some of the proofs that depend upon it, but is left tacit in others (for example in Theorems 10.4 and 11.3). However, this inconsistency is not peculiar to the present author.

The book is lavishly provided with examples, which range from simple illustrative problems to theorems (with hints of proofs) that extend or broaden the theory given in the text. There is a bibliography (26 titles), an index of postulates, an index of symbols, and a general index. The typography is excellent and there are hardly any misprints.

H. P. Mulholland (Birmingham).

Tomita, Minoru. Measure theory of complete Boolean algebras. Mem. Fac. Sci. Kyûsyû Univ. A. **7**, 51–60 (1952).

Results on the connection between a Boolean algebra B and its Stone space X; e.g., B is complete if and only if X is extremely disconnected, and a positive measure on B has weird topological properties on X (i.e., every measurable function is equal to a continuous function except on a nowhere dense set). The paper contains nothing new.

P. R. Halmos (Chicago, Ill.).

Orihara, Masae, and Tsuji, Kazô. Measures in non-separable topological spaces. Mem. Fac. Sci. Kyûsyû Univ. A. **6**, 167–172 (1952).

In this paper, an attempt is made to generalize a theorem of Marczewski and Sikorski [Colloquium Math. 1, 133–139

(1948); these Rev. 10, 23]. Let X be a topological space, $\mathfrak{B}$ the Borel sets in X (=the smallest σ-algebra of subsets of X which contains all closed sets). One considers countably additive, non-negative, σ-finite measures μ defined on $\mathfrak{B}$. A cardinal number $\mathfrak{m}$ has measure 0 if there exists no countably additive measure defined on all subsets of a set of cardinal number $\mathfrak{m}$ which assumes exactly the two values 0 and 1 and vanishes for points. A partial determination of cardinal numbers of measure 0 has been carried out by S. Ulam [Fund. Math. 16, 140–150 (1930)]. Marczewski and Sikorski prove that if X is metric and contains a dense subset D such that $\bar{\bar{D}}$ has measure 0, then for every measure μ of the type described, there is an open subset N of X such that $\mu(N)=0$ and N' is separable. The present authors seek to prove this result for all regular spaces having at each point a complete family of neighborhoods completely ordered by set-inclusion. Their paper is very closely modeled on the paper of Marczewski and Sikorski [loc. cit.], and their proof fails at line 25 of page 170, where a very special property of metric spaces, essential in the argument of Marczewski and Sikorski, is ignored for the case being considered. The proof of Lemma 2, page 168, is also faulty.
E. Hewitt (Seattle, Wash.).

Králik, D. Bemerkungen über nicht-messbare Punktmengen. Publ. Math. Debrecen 2, 229–231 (1952).

A known method applied to prove a well-known result: if a set in a Euclidean space is not of Lebesgue measure zero, then it has a non-measurable subset. *P. R. Halmos.*

Kinokuniya, Yoshio. On continuum. Mem. Muroran Coll. Tech. 1, no. 3, 313–317 (1952).

The author defines the measure of a point $x \in [0, 1]$ as $1/2^{\aleph_0}$, a number which he confesses is hard to define, and considers a number of similar devices. *E. Hewitt.*

Hildebrandt, T. H. Integration in abstract spaces. Bull. Amer. Math. Soc. 59, 111–139 (1953).

The author gives a summary of the recent progress of the theory of integration in abstract spaces, with emphasis on Lebesgue integration, in one case in the direction of linear spaces, and in the other using the order character of some linear spaces.

The content of the first part is: (1) the Riemann integral, (2) the Bochner integral, (3) the Dunford (first) integral, (4) other approaches to the Bochner integral, (5) the Birkhoff integral, (6) bilinear integrals, (7) the Price integral, (8) the Gelfand-Pettis integral, and (9) Phillips-Rickart generalizations. The content of the second part is: (1) the Daniell integral, (2) measurability of functions and sets in Daniell integrals, (3) extensions of Daniell integrals, (4) special cases, (5) Carathéodory theory of integrals over spaces without points, and (6) an additional approach to abstract integrals.

Finally the author writes that "... integration is essentially a linear process operating on a linear space. As suggested by the Lebesgue postulates, one abstracts and considers integration as a transformation from one space to another preserving certain properties, such as linearity, order, boundedness and convergence. Of course, one could make the rash statement that an integral is a linear 'continuous' form, a statement which is quite in keeping with recent trends (A. Weil, L. Schwartz). This is a logical sequel to the expression of the most general linear form on continuous functions on a finite interval as a Stieltjes integral or that on the space of bounded functions as a generalized integral. A second notion which plays a prominent role is that of extension. The value of an 'integral' is known for a certain group of elements, subset of a larger group. One tries to extend to a larger group preserving certain properties, obtaining other properties for the larger group, ...".
S. Izumi (Tokyo).

Krickeberg, Klaus. Zur Theorie des oberen und unteren Integrals. Math. Nachr. 9, 86–128 (1953).

A presentation of the theory of integration of real-valued functions by positive measures on a σ-ring of sets. The approach is via countable partitions and the upper and lower sums they define; emphasis is placed on an abstinence from finiteness (and σ-finiteness) conditions whenever possible, and on the fact that integrability can be defined without defining measurability first. The techniques and results are not new. *P. R. Halmos* (Chicago, Ill.).

See the corresponding note p. 1278

Pí Calleja, Pedro. On the concept of the integral. Revista Soc. Cubana Ci. Fis. Mat. 3, 8–23 (1953). (Spanish)

For the preceding part and reference to earlier parts see these Rev. 14, 28.

Čelidze, V. G. Double Denjoy integrals. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 155–242 (1947). (Russian. Georgian summary)

The extensions of the Denjoy integral to functions of two variables, given by Looman, Krzyżański, Kempisty, and Romanowski, deal with the so-called $\mathfrak{D}$-integral in the restricted sense, so that the integrand is the ordinary derivative of its integral almost everywhere (Kempisty considers regular derivatives, Looman considers strong derivatives). The author develops the corresponding theory of double Denjoy integrals in the wide sense, so that in his case the integrand is the approximate derivative of the integral a.e.

The function $F(x, y)$ defined on a 2-dimensional interval R_0 is said to be absolutely continuous (AC) on the set $E\subset R_0$ if given $\epsilon>0$ there is an $\eta>0$ such that if $r_1, \cdots, r_n$ is a system of non-overlapping intervals such that each r_i has two opposite end-points belonging to E, then $\sum|r_i|<\eta$ implies $\sum|\Delta(F, r_i)|<\epsilon$, where $\Delta(F, r)$ is the increment of F on r. If the r_i are required to have all their end-points belonging to E, then F is said to be AC in the wide sense on E. $F(x, y)$ is ACG on E if $E=\sum E_n$ and F is AC on each E_n. The author proves the following results: If $F(x, y)$ is continuous on R_0 and ACG in the wide sense on the set $E\subset R_0$, then F has an approximate derivative $D_{ap}F(x, y)$ a.e. on E. If $F(x, y)$ is continuous and ACG on R_0, then $D_{ap}F(x, y)\geqq 0$ a.e. on R_0 implies $\Delta(F, R_0)\geqq 0$. There are functions AC in the sense of the author and not in the sense of Kempisty. If the regular derivatives $\bar{D}F(x, y)$ and $\underline{D}F(x, y)$ are finite on R_0 outside of a denumerable set of lines parallel to the axes, then F is ACG on R_0.

A function $f(x, y)$ is said to be ($\mathfrak{D}$)-integrable on R_0 if there exists a continuous ACG function $F(x, y)$ such that $D_{ap}F(x, y)=f(x, y)$ a.e. The properties of the ordinary Denjoy integral hold for this double Denjoy integral. The paper contains many other results too numerous to be quoted here. *M. Cotlar* (Buenos Aires).

Brunk, H. D., and Ewing, G. M. Line integral approximation of double integrals. Proc. Amer. Math. Soc. 4, 287–295 (1953).

Wilkins [Bull. Amer. Math. Soc. 55, 191–192 (1949); these Rev. 10, 516] and the reviewer [Proc. Amer. Math.

Soc. **2**, 706–709 (1951); these Rev. **13**, 215] proved that if the region R is a circle with center at the origin, if $F(x, y)$ is continuous in R and if γ is the spiral $\theta=\alpha r$, then $(2\pi)^{-1}\int\int_R F(x, y)dxdy=\lim_{\alpha\to\infty}\alpha^{-1}\int_\gamma F(x, y)ds$. The authors extend this result, by proving similar theorems, valid for regions R of a quite general type. Specifically, let $\alpha>0$, $a>0$; let R be a bounded, measurable subset of the plane, contained in the circle of radius a; set $g(r, \theta)=f(r, \theta)h(r)$, with $f(r, \theta)$ continuous on the closure of R and periodic in θ, and with $h(r)\geqq 0$ almost everywhere and summable over $[0, a]$. Define the spiral Γ_α by $\theta=\alpha\int_0^r h(u)du$ and the sequence $\{r_i\}$ by $2i\pi=\alpha\int_0^{r_i}h(u)du$. Then, if

$$I=\frac{1}{2\pi}\int_R\int g(r, \theta)drd\theta,$$

$$I^*(\alpha)=\frac{1}{\alpha}\int_{\Gamma_\alpha\cap R} f(r, \theta)d\theta=\int_{\Gamma_\alpha\cap R} g(r, \theta)dr$$

and R satisfies some rather mild conditions (e.g., is bounded by a finite set of rectifiable simple closed curves), it is shown that $I=\lim_{\alpha\to\infty}I^*(\alpha)$. The proof uses Fubini's theorem, in order to find an upper bound for $|I-I^*(\alpha)|$. Some special results are obtained in case $f(r, \theta)$ does not depend on r, and a necessary and sufficient condition is given in order to have $I=I^*(\alpha)$, without passage to the limit.

E. Grosswald (Philadelphia, Pa.).

Kinokuniya, Yoshio. A constructive study of the functions at the points of discontinuity in the theory of Stieltjes integration. Mem. Muroran Coll. Tech. **1**, no. 3, 319–323 (1952).

If $G(x)$ is of bounded variation, then a point of discontinuity ξ of $G(x)$ contributes the term $f(\xi)[G(\xi+0)-G(\xi-0)]$ to the Stieltjes integral $\int f(x)dG(x)$. This note considers conditions under which if $f_1(x, \delta)$ is continuous and passes through $(\xi-\delta, f(\xi-0))$, $(\xi, f(\xi))$, and $(\xi+\delta, f(\xi+0))$, and $G_1(x, \delta)$ through $(\xi-\delta, G(\xi-0))$, $(\xi, G(\xi))$, and $(\xi+\delta, G(\xi+0))$, we have

$$\lim_{\delta\to 0}\int_{\xi-\delta}^{\xi+\delta} f_1(x, \delta)dG_1(x, \delta)=f(\xi)[G(\xi+0)-G(\xi-0)].$$

Two cases in which the approaching functions are polygonal are considered. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Kuipers, L., and Meulenbeld, B. On real functions of n variables. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 490–497 (1952).

The authors continue their work in the theory of C-uniform distribution (for the definition see a previous paper of Meulenbeld, same Proc. **53**, 311–317 = Indagationes Math. **12**, 59–65 (1950); these Rev. **11**, 648; and a paper by Kuipers and Meulenbeld, same Proc. **52**, 1151–1157, 1158–1163 = Indagationes Math. **11**, 425–431, 432–437 (1949); these Rev. **11**, 423). The authors prove several theorems about the uniform distribution and non-uniform distribution of functions. We here state only one of their theorems: Let there be given a sequence F of n-dimensional intervals Q: $0\leqq S_u<t_u<T_u$, $u=1, 2, \cdots, n$, where $T_n-\phi_n$ and the volume of Q tends to infinity as Q runs through F. Further, assume that $f(t_1, t_2, \cdots, t_n)$ is defined in all the Q's and has a partial derivative with respect to t_n. Assume finally that $\lim_{t_n\to\infty}\partial f/\partial t_n=c\neq 0$ uniformly in $t_1, t_2, \cdots, t_{n-1}$. Then $f(t)$ is uniformly distributed in the intervals Q of F.

P. Erdős (Los Angeles, Calif.).

Eggleston, H. G. Some remarks on uniform convergence. Proc. Edinburgh Math. Soc. (2) **10**, 45–52 (1953).

Several sufficient conditions, most of them previously known, for uniform convergence of a sequence of real functions of a real variable are deduced. Typical of these is a theorem of Buchanan and Hildebrandt [Ann. of Math. (2) **9**, 123–126 (1908)]: If on an interval $[a, b]$ a sequence $\{f_n\}$ of monotonic functions converges pointwise to a continuous function f, then the convergence is uniform. The principal tool is the following theorem, apparently hitherto unrecorded. Let X_m $(m=1, 2, \cdots)$ be rectifiable arcs, of respective lengths l_m, all lying in the same bounded portion of Euclidean n-space. Let $\lim l_m=l$ exist, and let X be an arc of length l lying in the (topological) limit inferior of the sequence $\{X_m\}$. Then $X=\limsup X_m$. Some of the theorems are extended to the case of functions of several real variables.

T. A. Botts (Charlottesville, Va.).

Walsh, J. L. On continuity properties of derivatives of sequences of functions. Proc. Amer. Math. Soc. **4**, 69–75 (1953). See the corresponding note p. 1278

The author discusses the pth derivatives of convergent sequences of functions of a real variable. Let $f_n(x)\to f(x)$ in the interval I, and let the pth derivatives $f_n^{(p)}(x)$ exist at all points of I. The following theorems are proved: (1) If $f^{(p)}(x)$ also exists, then, given any neighbourhood U of a point $x_0\in I$ and any $\epsilon>0$, there is N (depending on U and ϵ) such that, for every $n>N$, there exists $x_n\in U$ for which $|f_n^{(p)}(x_n)-f^{(p)}(x_0)|<\epsilon$. (2) If each $f_n^{(p)}(x)$ possesses the modulus of continuity $\omega(\delta)$ in I, then $f^{(p)}(x)$ exists and has the modulus of continuity $\omega(\delta)$ in I. The author remarks that in this case the hypothesis in fact implies the uniform convergence of $f_n^{(p)}(x)$ and of all lower derivatives.

U. S. Haslam-Jones (Oxford).

Targonszky, G. I. Darstellung von Funktionen durch Kettenreihen. Publ. Math. Debrecen **2**, 286–289 (1952).

The following two theorems are proved. Let $f(x)$ be a real function differentiable in an interval J containing zero and strictly monotone increasing in J. If $f(0)=0$ and $|f(x)|>|x|$ when $x\neq 0$ and $x\in J$, then, for $x\in J$, $f(x)=\sum_{n=0}^\infty g_n(x)$ where $g_0(x)=x$, $g_1(x)=f^{-1}(f(x)-x)$ and $g_{n+1}(x)=g(g_n(x))$, $n=1, 2, 3, \cdots$. Let $f(x)$ be a real function with a continuous derivative in an interval of real numbers containing zero. Suppose $f'(0)\neq 0$ and select any real number p such that

$$|p|>\max_J\left|\frac{x}{f(x)-f(0)}\right|,\quad \operatorname{sgn} p=\operatorname{sgn} f'(0).$$

Then there exists an interval J containing zero in which $f(x)$ is strictly monotone and in which $f(x)=f(0)+p^{-1}\sum_{n=0}^\infty g_n(x)$ where $g_0(x)=x$, $g_1(x)=f^{-1}(f(x)-x/p)$ and $g_{n+1}(x)=g(g_n(x))$, $n=1, 2, 3, \cdots$. *W. R. Utz* (Columbia, Mo.).

R. Salinas, Baltasar. Note on the asymptotic behavior of the reiterated application of a sequence of functions. Gaceta Mat. (1) **4**, 81–90 (1952). (Spanish)

A number of results are established, of which the following is the most general: Let $\{f_n(x)\}$ $(n=1, 2, \cdots)$ be real functions on the open interval $(0, \alpha)$, with $f_n(x)\neq 0$ for $0<x<\alpha$ and $n=1, 2, \cdots$; let $\{a_n\}$ be positive numbers with $A_n=\sum_{k=1}^n a_k\to\infty$ and $a_n/A_n\to 0$; and let $\varphi(x)$ be a positive continuous function on $(0, \infty)$ such that $x^s\varphi(x)$ is increasing for some $s<1$, and such that: (i) for each X in $0<X<\alpha$,

$$\epsilon(X)=\inf_{\substack{0<x\leqq X\\ n=1,2,\cdots}}\left\{\frac{1}{a_n\varphi(A_{n-1}x)}\left[\frac{1}{f_n(x)}-\frac{1}{x}\right]\right\}>0;$$

(ii) for each n, $f_n(x)=x-a_nx^2\varphi(A_{n-1}x)\theta_{n-1}(x)$ where $\theta_n(y/A_n)\to 1$ uniformly in y on each interval $(0, Y)$. Define the sequence $\{F_n(x)\}$ by $F_1(x)=f_1(x)$; $F_n(x)=f_n(F_{n-1}(x))$, $n>1$. Then $\lim_{n\to\infty} A_nF_n(x)=\lambda$ where λ is the root of the equation $\lambda\varphi(\lambda)=1$. *I. M. Sheffer* (State College, Pa.).

Bertolini, Fernando. Osservazioni sulle funzioni omogenee. Boll. Un. Mat. Ital. (3) **8**, 65–71 (1953).

Sambo, Alberto. Sulla derivazione sotto il segno di integrale. Rend. Sem. Mat. Univ. Padova **21**, 252–255 (1952).

The author observes that a theorem of Volpato [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 146–150 (1952); these Rev. **14**, 148], concerning differentiation under the integral sign, can be deduced from a result of Scorza-Dragoni [Rend. Sem. Mat. Univ. Padova **20**, 462–467 (1951); these Rev. **13**, 827]. *T. Radó.*

Cecconi, Jaurès. Sulla differenziabilità, nel senso di Stolz, di una funzione di più variabili. Ricerche Mat. **1**, 317–324 (1952).

Let $f(x, y)$ be a real-valued function in the closed unit square $Q: 0\leqq x\leqq 1, 0\leqq y\leqq 1$, which satisfies there a Lipschitz condition. By a classical theorem of Rademacher, the first partial derivatives f_x and f_y exist then almost everywhere in Q, and $f(x, y)$ is differentiable in the sense of Stolz (that is, $f(x, y)$ possesses a complete differential) almost everywhere in Q. Suppose that $f(x, y)$ is not only Lipschitzian but satisfies the following further condition: for almost every x in the interval $(0, 1)$, f_x and f_y are continuous functions of y, and for almost every y in the interval $(0, 1)$, f_x and f_y are continuous functions of x. The author proves then that $f(x, y)$ is differentiable in the sense of Stolz on a subset F of Q such that the set $E=Q-F$ is of "pseudo linear measure zero." This property is defined as follows. Let D^x denote, generically, a system of non-overlapping intervals in the unit interval of the x-axis. Each interval $x'\leqq x\leqq x''$ in D^x determines a rectangle $x'\leqq x\leqq x''$, $0\leqq y\leqq 1$. Let RD^x be the system of rectangles so associated with D^x, and let D^y, RD^y have analogous meanings relative to the y-axis. Finally, let $|D^x|$, $|D^y|$ denote the sum of the lengths of the intervals of the systems D^x, D^y, respectively. A set $E\subset Q$ is then of "pseudo linear measure zero" if for every positive integer n there exist systems D_n^x, D_n^y such that $|D_n^x|<1/n$, $|D_n^y|<1/n$, and E is covered by the rectangles of the systems RD_n^x and RD_n^y. Thus it appears that the additional assumption used by the author (see above) yields a conclusion which is stronger than that in the classical Rademacher theorem. The paper contains two more theorems of a similar character. These results will be applied, according to the author, in the study of the uniqueness of the solutions of partial differential equations. *T. Radó.*

Kreisel, G. Note on functional relationship. Math. Gaz. **37**, 18–20 (1953).

Let $u(x, y)$, $v(x, y)$ be real-valued functions in a closed region. In elementary texts on analysis, one finds the theorem that there exists a functional relation between u and v if and only if the functional determinant $\partial(u, v)/\partial(x, y)$ vanishes identically in the region. The author points out that difficulties arise if one permits the four partial derivatives u_x, u_y, v_x, v_y to vanish simultaneously at some points, and proposes a version of the concept of "functional relationship" which seems suitable as a basis for an adequate discussion of this topic in an elementary course in analysis. *T. Radó* (Columbus, Ohio).

***Comét, Stig. Conformal mapping and group automorphisms.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 122–129. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let $y_i=y_i(x_1, \cdots, x_n)$, $i=1, \cdots, n$, be real-valued functions of the real variables $x_1, \cdots, x_n$ with continuous first partial derivatives in a certain domain. It is assumed that the matrix $Y=\|\partial y_i/\partial y_j\|$ is non-singular, and the functions y_i are thought of as defining a transformation T. The author points out that the condition for T to be conformal can be written in the form $Y'Y=\lambda^2I$, where Y' is the transpose of Y, I is the unit matrix, and λ is a scalar function of $x_1, \cdots, x_n$. It follows that if A is any orthogonal (n by n) matrix, then $B=YAY^{-1}$ is also orthogonal. Thus Y defines an automorphism of the orthogonal group. Conversely, if Y has this property, then the mapping T is conformal. The author proposes to use this characterization of conformal mappings as a starting point to obtain more general classes of mappings. For example, let H be a subgroup of the orthogonal group. Then H gives rise to a class of transformations T characterized by the property that for every orthogonal matrix $A \varepsilon H$ the matrix YAY^{-1} is orthogonal (where again Y is the functional determinant of T). Further generalizations are obtained by replacing the orthogonal group by the symplectic group. The author considers several special cases, and in each case he indicates the type of equations that correspond to the Cauchy-Riemann equations of the classical $n=2$ case. *T. Radó* (Columbus, Ohio).

Theory of Functions of Complex Variables

Gaier, Dieter. Schlichte Potenzreihen, die auf $|z|=1$ gleichmässig, aber nicht absolut konvergieren. Math. Z. **57**, 349–350 (1953).

If the function (1) $f(z)=\sum a_nz^n$ is regular in the unit disc D, it maps each radius of D into a curve of length at most $\sum_1^\infty |a_n|$. The author uses this fact, together with a Jordan region one of whose boundary points is not finitely accessible from the interior, to establish the existence of a function (1), schlicht in D, regular on $|z|=1$ except at $z=1$, and with $\sum a_nz^n$ converging uniformly but not absolutely on $|z|=1$. As the author notes, the existence of such a function was established earlier by Erdös, Herzog, and the reviewer [Pacific J. Math. **1**, 75–82 (1951); these Rev. **13**, 335]. The present paper makes the contribution of solving the problem without the annoyance of cumbersome computations. On the other hand, the previous construction is not yet obsolete, because it leads to a function (1) that maps each radius of D into a curve of length less than 3.

G. Piranian (Ann Arbor, Mich.).

Freud, G. Über einen Reihentheoretischen Satz von Fejér. Acta Math. Acad. Sci. Hungar. **3**, 173–176 (1952). (Russian summary)

A function $f(z)=\sum a_nz^n$ which is regular and bounded in $|z|<1$ and for which $\sum n|a_n|^2<\infty$ is termed a Fejér function. For such functions, Fejér proved that the power series converge at all points $z=e^{i\theta}$ where the radial limit exists [Monatsh. Math. Phys. **28**, 64–76 (1917)]. If $f(z)$ is regular, bounded, and schlicht in $|z|<1$, it is a Fejér function. The author proves that the class of all Fejér functions forms a ring. In particular, if $\varphi_1, \cdots, \varphi_n$ are regular, schlicht, and bounded in $|z|<1$, then any polynomial in these is a Fejér function. *R. C. Buck* (Madison, Wis.).

Roux, Delfina. Sul comportamento delle serie di potenze sugli archi di regolarità. Ann. Mat. Pura Appl. (4) **33**, 255–262 (1952).

For any power series $f(x)=\sum a_n x^n$ with partial sums $f_n(x)$, a theorem of Fatou-Riesz ensures that $f_n(x)\to f(x)$ uniformly on every (closed) arc of regularity on the circumference $|x|=1$, provided $a_n\to 0$. The present note establishes two theorems giving sufficient conditions for $f_{n_k}(x)\to f(x)$, $0=n_0<n_1<n_2<\cdots$, as $k\to\infty$, uniformly upon any such arc. The conditions take account of the behavior of $|n_{k+1}-n_k|$ and of the location of the singular points. (From the author's summary.) *R. M. Redheffer.*

Herzog, Fritz, and Piranian, George. Sets of convergence of Taylor series. II. Duke Math. J. **20**, 41–54 (1953).

In a previous paper [same J. **16**, 529–534 (1949); these Rev. **11**, 91] the authors showed that if C denotes the circumference $|z|=1$ and M is any G_δ subset of C, then there is a series $f(z)=\sum a_n z^n$ which diverges for $z\in M$ and converges for $z\in C-M$. This is now extended by the adjunction of an arbitrary countable set to M. For an arbitrary countable set M, functions $f(z)$ are constructed having M as their set of divergence, and which in addition may be required to be schlicht and bounded in $|z|<1$, or regular in $|z|<1$ and continuous in $|z|\leq 1$. The boundary behavior of $f(z)$ may be still further specified with respect to the existence of radial limits. On the other hand, functions are constructed of each of these classes whose set of divergence is locally nondenumerable everywhere on C. Similarly, functions are constructed in each class whose Taylor series converge everywhere on C, but uniformly in no arc. Call a point $z_0\in C$ a point of unboundedness for $f(z)$ if $f(z)$ is defined on C but is unbounded on every arc containing z_0. A necessary and sufficient condition for M to be the set of unboundedness for some Taylor series converging on C is that M be closed and nowhere dense. It is also observed that many of the functions constructed here also provide examples of functions mapping $|z|<1$ onto sets which have inaccessible boundary points. [See A. W. Goodman, Proc. Amer. Math. Soc. **3**, 742–750 (1952); these Rev. **14**, 367.] *R. C. Buck* (Madison, Wis.).

Noble, M. E. On Taylor series with gaps. J. London Math. Soc. **28**, 197–203 (1953).

The following theorem is proved. Let the series $\sum_0^\infty a_n z^n$ and the function $f(z)$ represented by it satisfy the following two conditions with reference to the point $z_0=e^{i\theta_0}$. (1) $f(z)$ is bounded in some sector $|\theta_0-\arg z|<\delta_0$, $0<\rho<1$; and there exists a constant $A(\theta_0)$ such that, with $|\delta|\leq\delta_0$, the relation

$$\lim_{\delta\to 0}\int_{\theta_0-\delta}^{\theta_0+\delta}\left|\frac{f(\rho e^{i\theta})-A(\theta_0)}{\rho e^{i\theta}-e^{i\theta_0}}\right|d\theta=0$$

holds uniformly with respect to ρ $(0<\rho<1)$. (2) There exist increasing sequences $\{n_k\}$ and $\{N_k\}$ such that $a_n=0$ if $n_k<n\leq N_k$, and such that $\phi(n_k)=o(N_k-n_k)$, where $\phi(n)$ is the concave majorant of $\log(|a_n|+2)$. Then $\sum_0^{n_k}a_n z_0^n\to A(\theta_0)$. If condition (1) is satisfied uniformly on an arc C of the unit circle, the overconvergence is uniform on C.

The author notes that his theorem is similar to earlier results by Erdös and the reviewer [Duke Math. J. **14**, 647–658 (1947); these Rev. **9**, 232]. In the earlier results, condition (1) is replaced by the much stronger requirement that $f(z)$ be regular at z_0, and the gaps are required to be somewhat longer than is specified in (2); on the other hand, the earlier results include estimates on the order of approximation of the sequence $\{\sum_0^{n_k}a_n z^n\}$ to $f(z)$. *G. Piranian.*

Turán, Paul. On a property of lacunary power-series. Acta Sci. Math. Szeged **14**, 209–218 (1952).

The paper contains results on entire functions

$$f(z)=\sum_0^\infty a_n z^{\lambda_n}$$

with Fabry gaps $(n/\lambda_n\to 0)$ related to those of Pólya and Sunyer i Balaguer [cf. Sunyer i Balaguer, Collectanea Math. **2**, 129–174 (1949); these Rev. **12**, 489]. For such functions and sufficiently large r the inequality

$$M(r)^{1+\epsilon}\leq\{48\pi/(\beta-\alpha)\}M(2r)^{\epsilon}M(r,\alpha,\beta)$$

is established where $M(r,\alpha,\beta)$ is the maximum modulus of $f(z)$ on $|z|=r$ with $\alpha\leq\arg z\leq\beta$ and $M(r)=M(r,0,2\pi)$. A similar result for harmonic functions with lacunary Fourier series is obtained. The proof is novel and "elementary", exploiting an inequality previously given by the same author [these Rev. **9**, 80]. *A. J. Macintyre.*

Čakvetadze, S. S. Solution of a boundary problem of Haseman for several unknown functions. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 449–455 (1951). (Russian)

The problem considered is of the type studied first by Charles Haseman [Dissertation, Göttingen, 1907]. Let L be a simple, closed, suitably smooth contour in the complex plane; D^+ is the finite domain bounded by L; D^- is the complement of D^++L; $\alpha(t)$ is a function whose derivative is of class H on L and which transforms L one-to-one on itself; t and $\alpha(t)$ describe L in opposite directions. The problem is to find a piecewise analytic vector $\phi=(\phi_1,\cdots,\phi_n)$, of finite order at infinity, so that

$$\text{(1)}\quad \phi_k^+[\alpha(t_0)]=\sum_1^n G_{kj}(t_0)\overline{\phi_j^-}(t_0)+g_0(t)\quad(k=1,\cdots,n),$$

the coefficients being in H. The case $n=1$ has been studied (in an unavailable paper) by D. A. Kveselava [same Soobščeniya 7, no. 10 (1946)]. The author solves the case when $n>0$, following a method used by N. P. Vekua [Systems of singular integral equations . . . , Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; these Rev. **13**, 247]. In matrix notation (1) takes the form (2) $\phi^+[\alpha(t_0)]=G(t_0)\overline{\phi^-}(t_0)+g(t_0)$ (on L). The homogeneous equation (I) (2; $g=0$) is handled with the aid of a certain related Fredholm integral equation; the latter leads either to a solution of (I) or to a solution, vanishing at infinity, of the adjoint equation $\psi^+(t_0)=\overline{G^{-1}}(t_0)\overline{\psi^-}[\alpha(t_0)]$. All the solutions of (I), of finite order at infinity, are given explicitly with the aid of a canonical system of solutions of (I). Necessary and sufficient conditions are found in order that (2) should have solutions vanishing at infinity. These results can be extended to the case of a finite number of disjoint contours, limiting a connected domain. On the basis of the above, a study can be made of a corresponding system of integral equations in the sense of principal values. *W. J. Trjitzinsky* (Urbana, Ill.).

Sunyer i Balaguer, F. Values of entire functions represented by gap Dirichlet series. Proc. Amer. Math. Soc. **4**, 310–322 (1953).

For the Dirichlet series $\sum_0^\infty a_n\exp(-\lambda_n s)$ with

$$\liminf(\lambda_{n+1}-\lambda_n)=h>0$$

representing an integral function $f(s)$ of finite order ρ in the

sense of Ritt [Amer. J. Math. **50**, 73–86 (1928)] the following theorem is proved. Let Y be a given positive number greater than π/ρ. Then there exist positive numbers Δ and B depending on Y, ρ, h such that $\limsup n(\sigma, f-a)/V(\sigma) \geqq B$ for all a without exception providing the mean upper density [Mandelbrojt, Ann. Sci. Ecole Norm. Sup. (3) **63**, 351–378 (1947); these Rev. **9**, 229, 735] does not exceed Δ. Here $n(\sigma, f-a)$ represents the number of zeros of $f(s)$ in $|\Im(s)| \leqq \frac{1}{2}Y$ with $\sigma \leqq x \leqq 0$ and $V(\sigma)$ is a proximate order defined and discussed in the present paper. The investigation is extended to functions of infinite order.

A. J. Macintyre (Aberdeen).

Vermes, P. An interpolation problem for integral functions. J. Analyse Math. **2**, 150–159 (1952).

J. M. Whittaker [Sur les séries de base de polynomes quelconques, Gauthier-Villars, Paris, 1949; these Rev. **11**, 344] proposed the problem of finding an entire function $h(z)$ such that $h^{(n)}(1) = f^{(n)}(1)$ for $n = p_j$, $h^{(n)}(0) = f^{(n)}(0)$ for $n = q_j$, where $\{p_j\}$, $\{q_j\}$ are given sequences and f and g are given entire functions. Whittaker gave a necessary condition for $h(z)$ to be uniquely determined by the data. The author now applies the theory of infinite systems of linear equations to obtain some sufficient conditions for the problem to have a solution. Let $f(z) = \sum f_k z^k/k!$, $g(z) = \sum g_k z^k/k!$; one of the author's conditions is that Whittaker's necessary condition is satisfied and $|f_{q_j} - g_{q_j}| \leqq \beta_j$, where $\{\beta_j\}$ is a certain sequence depending on $\{p_j\}$ and $\{q_j\}$ but not on $f(z)$ and $g(z)$. Similar results involve the Taylor coefficients for $z=1$ or $z=\frac{1}{2}$. Another group of results restricts the sequences more and the functions less.

R. P. Boas, Jr.

Berghuis, J. A class of entire functions used in analytic interpolation. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 468–473 (1952).

Let $F_k(t, u) = \sum (\sin v/v)^k \exp(-tv/\pi)$, where $v = \pi(u+n)$ and the sum is for all n from $-\infty$ to ∞. The author studies the properties of $F_k(t, u)$, including certain integral and product representations. It is shown for $k=1$ and 2 that $1/F_k(t, u)$ has an expansion of the form $\sum_0^\infty c_n(t)(2\sin \pi u)^{2n}$ where $c_n(t)$ is positive, a result which was conjectured by I. J. Schoenberg [Quart. Appl. Math. **4**, 112–141 (1946); these Rev. **8**, 55] and also proved by him for the case $t=0$.

P. W. Ketchum (Urbana, Ill.).

Wilson, R. Directions of strongest growth of the product of integral functions of finite order and mean type. J. London Math. Soc. **28**, 185–193 (1953).

Let ρ_i, k_i, and $k_i(\theta)$ denote, respectively, the order, the type, and the indicator function of the entire function $F_i(z)$ of finite order and mean type $(i=1, 2)$. Directions of strongest growth (DSG) of $F_i(z)$ are the values of θ for which $k_i(\theta)$ attains its maximum k_i. The author studies the product $H(z) = F_1(z)F_2(z)$. (1) If $\rho_1 > \rho_2$, then $H(z)$ is of order ρ_1 and has indicator $k_1(\theta)$. (2) Let $\rho_1 = \rho_2 = 1$. Let $f_i(z)$, $h(z)$ denote the Borel (Laplace) transforms of $F_i(z)$, $H(z)$. By a result of Pólya [Math. Z. **29**, 549–640 (1929)] the DSG of $H(z)$ are the same as the directions of the radii joining the origin to the singular points γ on the circle of convergence of $z^{-1}h(z^{-1}) = \sum c_n z^n$. By another known result these points γ are of the form $\alpha_1 + \alpha_2$, α_i a singular point of $z^{-1}f_i(z^{-1})$. (3) The author obtains a similar result in the general case $\rho_1 = \rho_2 = \rho > 0$ by introducing the quasi-entire functions $F_i(z^{1/\rho})$. (4) The author finally expresses the DSG of $H(z)$ in the case $\rho_1 = \rho_2$ in terms of those of the $F_i(z)$ after developing a suitable extension of the concept of DSG. Such an extension is necessary since the above points α_i do not have to lie on the circles of convergence of the series $\sum a_{i,n} z^n = z^{-1} f_i(z^{-1})$ and therefore do not necessarily correspond to (ordinary) DSG of $F_i(z)$.

J. Korevaar.

Boas, R. P., Jr. Two theorems on integral functions. J. London Math. Soc. **28**, 194–196 (1953).

D'après Valiron [Ann. Fac. Sci. Univ. Toulouse (3) **5**, 117–257 (1914)] et Pólya [Math. Ann. **88**, 169–183 (1923)], si $f(z)$ est une fonction entière d'ordre non entier, $M(r)$ le maximum du module, et $n(r)$ le nombre des zéros dans $|z| \leqq r$, on a $\limsup_{r\to\infty} n(r)/\log M(r) > 0$. Shah a montré [J. London Math. Soc. **15**, 23–31 (1940); ces Rev. **1**, 307, 400] que dans le cas de l'ordre entier, on a

$$\limsup_{r\to\infty} n(r)\phi(r)/\log M(r) = \infty$$

pour toute fonction croissante $\phi(r)$ telle que (*) $\int_1^\infty [t\varphi(t)]^{-1}dt$ converge. L'auteur établit ces deux propositions par une méthode plus rapide, indépendante de la notion d'ordre précisé de Valiron ou des théorèmes de Pólya, en supposant seulement dans le cas de l'ordre entier que $\phi(r)$ est positive et que l'intégrale (*) existe.

G. Valiron (Paris).

***MacLane, G. R. Riemann surfaces and asymptotic values associated with real entire functions.** Rice Inst. Pamphlet. Special Issue. The Rice Institute, Houston, Texas, 1952. i+93 pp.

The functions $f(z)$ discussed are of finite order and such that all zeros of the derivative $f'(z)$ are real and positive. The pamphlet gives a detailed development of results announced in 1950 [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 1, Amer. Math. Soc., Providence, R. I., 1952, pp. 396–397] generalising earlier work by the same author [Trans. Amer. Math. Soc. **62**, 99–113 (1947); these Rev. **9**, 85]. A substantial part of the work is devoted to the theory of conformal representation of variable domains (Carathéodory kernels) and to the approximation of transcendental by rational functions.

A. J. Macintyre (Aberdeen).

Gel'fond, A. O. Linear differential equations of infinite order with constant coefficients and asymptotic periods of entire functions. Amer. Math. Soc. Translation no. **84**, 31 pp. (1953).

Translated from Trudy Mat. Inst. Steklov. **38**, 42–67 (1951); these Rev. **13**, 929.

Goodman, A. W. The rotation theorem for starlike univalent functions. Proc. Amer. Math. Soc. **4**, 278–286 (1953).

S désigne la classe des fonctions $F(z) = z + \sum_2^\infty a_n z^n$ univalentes dans $|z| < 1$ qui représentent tout cercle $|z| \leqq r < 1$ dans une région étoilée par rapport à $z=0$; Σ est la classe des fonctions $\Phi(\zeta) = \zeta + \sum_0^\infty b_n \zeta^{-n}$ holomorphes sauf à l'infini, univalentes pour $|\zeta| > 1$ qui représentent tout cercle $|\zeta| \geqq \rho > 1$ sur une région étoilée par rapport à $\zeta = 0$. L'auteur donne des bornes supérieures pour

$$R_1 = R_1(z) = \arg F'(z), \qquad R_2 = R_2(\zeta) = \arg \Phi'(\zeta),$$
$$T_1 = T_1(z) = \arg F(z) - \arg z, \quad T_2 = T_2(\zeta) = \arg \Phi(\zeta) - \arg \zeta,$$

et montre que les bornes obtenues pour R_1, T_1, T_2 sont les meilleures possibles. Il retrouve ainsi et complète par des méthodes plus rapides des résultats antérieurs, notamment de Stroganoff [Trudy Fiz.-Mat. Inst. Steklov. **5**, 247–258 (1934)] et Birnbaum [Studia Math. **1**, 159–190 (1929)]. Il utilise entre autres des résultats de Robertson et la trans-

formation de Schwarz-Christoffel. Parmi ses énoncés, signalons les suivants: Si $F(z) \in S$ et si $0<|z|=r<1$,

$$|R_1(z)| \leqq \arc\tan \frac{r\sin\theta_0}{1+r\cos\theta_0}+3\arc\tan\frac{r\sin\theta_0}{1-r\cos\theta_0},$$

où $0<\theta_0<\pi/2$ et $\cos\theta_0=[r^2-1+(1+3r^4)^{1/2}]/2r$;

$$|T_1(z)| \leqq 2 \arcsin r.$$

Si $\Phi(\zeta) \in \Sigma$ et si $|\zeta|=\rho>1$,

$$|T_2(\zeta)| \leqq 2 \arcsin \rho^{-1}.$$

G. Valiron (Paris).

Rahmanov, B. N. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) 88, 413–414 (1953). (Russian)

A number of theorems are stated without proof. A typical result is the following. Let $\varphi_k(z)$ be univalent in $|z|<1$, $\varphi_k(0)=0$, $\varphi_k'(0)=1$, $k=1, 2, \cdots, n$. Then $\sum_{k=1}^{n}\varphi_k(z)$ is convex in $|z|<.188$. Theorem 5 of this paper was proved earlier by A. Kobori [Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. 17, 171–186 (1934)]. *A. W. Goodman.*

Krasnovidova, I. S., and Rogožin, V. S. A sufficient condition for univalency of the solution of an inverse boundary problem. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 151–153 (1953). (Russian)

Let $\varphi(s)$ and $\psi(s)$ be real differentiable functions of period l such that $\varphi^2(s)+\psi^2(s)=1$ and such that $0\leqq s_1<s_2<l$ implies that $|\varphi(s_1)-\varphi(s_2)|+|\psi(s_1)-\psi(s_2)|\neq 0$. If s is considered as arc length on a closed curve L, $w=\varphi+i\psi$ maps L onto the boundary of the unit circle $w=e^{i\theta}$. It is proved that if $\ln[\varphi'(s)^2+\psi'(s)^2]$ satisfies a Lipschitz condition with constant $\pi/\ln 4$ as a function of θ, then there is a simple domain D with boundary L and a function $w(z)$ analytic in D and mapping D onto $|w|<1$ in such a way that on L, $w(z)=\varphi(s)+i\psi(s)$. In the proof the authors rediscover a theorem initially due to Noshiro [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 2, 129–155 (1934)] and Warschawski [Trans. Amer. Math. Soc. 38, 310–340 (1935)], and extended by Herzog and Piranian [Proc. Amer. Math. Soc. 2, 625–633 (1951); these Rev. 13, 223]. *A. W. Goodman.*

Tammi, Olli. On certain combinations of the coefficients of schlicht functions. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 140, 13 pp. (1952).

For functions $f(z)$ schlicht, or analytic, in the unit circle, the author obtains coefficient estimates and distortion theorems which involve the total variation of $\arg zf'(z)$ or of $\log|f'(z)|$. *P. R. Garabedian* (Stanford, Calif.).

Suetin, P. K. Faber polynomials for regions with nonanalytic boundaries. Doklady Akad. Nauk SSSR (N.S.) 88, 25–28 (1953). (Russian)

Let $w=\Phi(z)$ be a univalent function with normalization $\Phi(\infty)=\infty$, $\Phi'(\infty)>0$, which maps the exterior region determined by a rectifiable Jordan curve Γ onto $|w|>1$, and let $z=\Psi(w)$ be the inverse function of $\Phi(z)$. The Faber polynomials $\Phi_n(z)$ associated with Γ are then defined as the terms involving the non-negative powers of z in the Laurent expansion of $[\Phi(z)]^n$ about $z=\infty$. Given a function $f(z)$, analytic in the interior region G determined by Γ, then under what conditions can $f(z)$ be expanded into a series of Faber polynomials which converges uniformly in G. Faber has shown that uniform convergence follows if nothing is required of $f(z)$ except analyticity, and Γ is analytic, or, if nothing else is required of Γ, and $f(z)$ is analytic in the closure of G [Math. Ann. 57, 389–408 (1903); 64, 116–135 (1907); J. Reine Angew. Math. 150, 79–106 (1920)]. The author finds conditions intermediate between the two extreme cases shown by Faber, e.g., if $f(z)$ is bounded in G and if Γ is such that (a) $\Psi'(w)$ is of class H_1 on $|w|=1$ and (b) $\int_{|w|=1}|\Psi'(w)|\log^+|\Psi'(w)||dw|<\infty$, then $f(z)$ can be expanded into a uniformly convergent series of Faber polynomials in G. *A. J. Lohwater* (Ann Arbor, Mich.).

Kuroda, Tadashi. On the uniform meromorphic functions with the set of capacity zero of essential singularities. Tôhoku Math. J. (2) 3, 257–269 (1951).

Let E be a nonempty, bounded, and closed set of capacity zero in the z-plane, and let $w=f(z)$ be single-valued and meromorphic in the complement of E and have an essential singularity at every point of E. Such functions are said to belong to the class $\mathfrak{F}$. Denote by $u(z)$ Evans' potential, which is harmonic at every finite point not belonging to E and tends to $+\infty$ when z tends to any point of E, and denote by $v(z)$ a harmonic function conjugate to $u(z)$.

The author extends a number of earlier results. In particular, the following extension is obtained of a result of Tsuji's [same J. (2) 3, 7–12 (1951); these Rev. 12, 692; see also G. af Hällström, Acta Acad. Aboensis 12, no. 8 (1940); these Rev. 2, 275], which is itself an extension of Ahlfors' distortion theorem. Let D be a simply connected domain in the z-plane and let E be a bounded, closed set of capacity zero on its boundary Γ. Let C_r be the level curve $e^{u(z)}=r$, where $u(z)$ is Evans' potential, and put $\theta_r=D\cap C_r$. If D is mapped conformally on $|w|<1$ by the function $w=f(z)$, then for all sufficiently large values of r, the image λ_r of θ_r in $|w|<1$ can be enclosed in a finite number of circles $k_r^{(i)}$ $(i=1, 2, \cdots, n=n(r))$ which cut $|w|=1$ orthogonally, such that the sum of their radii is less than

$$\text{const.}\times\exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right),$$

where r_0 is a certain positive number, $r_0<kr<r$, and $\theta(r)=\int_{\theta_r}dv(z)$. A consequence of this theorem is then the following theorem of Phragmén-Lindelöf type. Let $w=f(z)$ be regular in the simply connected region D and suppose that $\limsup_{z\to\zeta}|f(z)|\leqq 1$ at all points ζ of Γ not belonging to a certain bounded closed set E of capacity zero lying on Γ. Let $M(r)=\max_{z\in\theta_r}|f(z)|$ and let $\theta(\leqq 2\pi)$ denote the least upper bound of $\theta(r)$ for all sufficiently large r. If $\liminf_{r\to\infty} r^{-\pi/\theta}\log M(r)=0$, then $|f(z)|\leqq 1$ throughout D. The author defines a notion of order for functions of class $\mathfrak{F}$ and extends a result of Tsuji's [Jap. J. Math. 19, 139–154 (1944); these Rev. 8, 508] on the relation between the order of such functions and the singularities of the inverse function. Further theorems are also obtained. *W. Seidel.*

Collingwood, E. F. Sufficient conditions for reversal of the second fundamental inequality for meromorphic functions. J. Analyse Math. 2, 29–50 (1952). (Hebrew summary)

Ce mémoire est une suite de celui de l'auteur dans les Trans. Amer. Math. Soc. 66, 308–346 (1949) [ces Rev. 11, 94]. Les principaux résultats ont été donnés dans deux notes récentes [C. R. Acad. Sci. Paris 235, 1182–1184, 1267–1270 (1952); ces Rev. 14, 460, 461]. Nous nous bornerons donc à signaler ici la notion introduite par l'auteur de centre de ramification d'ordre non borné de la surface de Riemann S_f décrite par les valeurs d'une fonction

méromorphe $f(z)$. La limitation

$$\liminf_{r\to R} (N_1(r)/T(r))<2$$

implique l'existence d'au moins un tel centre. Collingwood pense que la valeur 1 pour le premier membre de cette inégalité serait la bonne limite impliquant l'existence de deux au moins de ces centres. *G. Valiron* (Paris).

Walsh, J. L. An interpolation series expansion for a meromorphic function. Trans. Amer. Math. Soc. **74**, 1–9 (1953).

Let $f(z)$ be meromorphic for $|z|<\infty$, let its finite poles belong to the sequence $\alpha_k\to\infty$ $(k=1, 2, \cdots)$, let the sequence $\beta_r\neq\alpha_k$ be uniformly limited, and let

$$r_n(z)=a_1/(z-\alpha_1)+\sum_{k=1}^{n} a_{k+1}(z-\beta_1)\cdots(z-\beta_k)/\{(z-\alpha_1)\cdots(z-\alpha_{k+1})\},$$

where the a_k are found by formal interpolation to $f(z)$ in the points β_k. Then $r_n(z)\to f(z)$ uniformly on any closed limited set containing no α_k [cf. Walsh, Interpolation and approximation . . . , Amer. Math. Soc. Colloq. Publ., v. 20, New York, 1935, §8.10, Corollary 2]. Here the author deals with the case when $f(z)$ is meromorphic merely in the deleted neighbourhood of a finite point O, the origin, say. His primary method is the use of uniformly distributed points [cf. loc. cit., §§8.7, 8.8]. He proves: Let C_0 and C_1 be Jordan curves; O in R_1, the interior of C_1; C_1 interior to C_0; and let the α_k' be disjoint from C_1, $\alpha_k'\to O$ $(k\to\infty)$. Then there exists a sequence α_k, consisting of the α_k' and of points of C_0, and a set β_k on C_1; such that, whenever $f(z)$ is analytic in R_1+C_1 except perhaps at O and at poles in α_k', $r_n(z)\to f(z)$ uniformly on any closed set in R_1+C_1 which excludes O and the α_k. He uses the function $U(z)$ which is harmonic in the annulus bounded by C_0 and C_1, continuous in the closed region, and $\equiv 0$ on C_0, $\equiv 1$ on C_1; and $g(z)$, the Green function for R_1 with pole at O. He introduces points which, in the sense of Weyl, are uniformly distributed with respect to the conjugates of $U(z)$ and $g(z)$, respectively: α_k'' on C_0, β_k'' on C_1; and β_k' on C_1. The α_k are obtained from the sets α_k' and α_k'', the β_k from the β_k' and β_k'', and their precise choice is discussed. It is a remarkable fact, new in the theory of interpolation, that the relative proportion of the points α_k' in the mixed set α_k is zero.

The $r_n(z)$ need not converge to $f(z)$ outside C_1, as is shown for $f(z)=1/z$. In Theorem 2, C_0 is the circle $|z|=1$, C_1: $|z|=r_1<1$, and specific formulae for the α_k and β_k are dealt with. In Theorem 3 a related problem is solved for a function $f(z)$ which is analytic in a circular annulus.

H. Kober (Birmingham).

Walsh, J. L., and Elliott, H. Margaret. Degree of approximation on a Jordan curve. Proc. Nat. Acad. Sci. U. S. A. **38**, 1058–1066 (1952).

In a former paper [Trans. Amer. Math. Soc. **71**, 1–23 (1951); these Rev. **13**, 451] H. M. Elliott has studied the relation between general moduli of continuity and problems of approximation. Here the method is used to generalise and to unify known results, mainly on the degree of approximation. Incidentally, the classical theory, developed by de la Vallée Poussin, on trigonometric approximation to periodic functions is generalised. There is a remarkable auxiliary theorem, a generalisation of Privaloff's well-known result: If $f(\theta)$, with period 2π, is continuous with modulus of continuity $\omega(\delta)$, $\int_0^h t^{-1}\omega(t)dt$ exists, $h>0$, and $g(\theta)$ is conjugate to $f(\theta)$, then

$$|g(\theta+h)-g(\theta)|\leq\omega_0(h)$$
$$\equiv A_1\left(\int_0^h\frac{\omega(t)}{t}dt+h\int_h^\pi\frac{\omega(t)}{t^2}dt\right)\quad(A_1=A(f)).$$

It yields new results for $\omega(t)=t^\alpha|\log t|^\beta$ $(0<\alpha\leq 1, \beta>0)$, etc. Again the theory of the components of a function $f(z)$ defined on an analytic Jordan curve C [see J. L. Walsh: (I) J. Math. Pures Appl. (9) **31**, 221–244 (1952); (II) Trans. Amer. Math. Soc. **73**, 447–458 (1952); these Rev. **14**, 547, 630] is used and generalised. Theorem 6. If $f^{(k)}(z)$ is continuous on C with modulus of continuity $\omega(\delta)$, $\omega_0(\delta)$ (see above)$\to 0$ as $\delta\to 0$, and $f_\nu(z)$ are the components of $f(z)$; where z is inside C for $\nu=1$, outside for $\nu=2$; then the boundary values of the $f_\nu(z)$ on C have moduli of continuity less than $\omega_0(\delta)$. If C contains $z=0$ in its interior, then there exist polynomials $p_n(z)$, $P_n(1/z)$, and $\pi_n(z, 1/z)=\sum_{k=-n}^{n}\alpha_{nk}z^k$ such that

$$|f_1(z)-p_n(z)|\leq A_2\omega_0(1/n)n^{-k},$$
$$|f_2(z)-P_n(1/z)|\leq A_3\omega_0(1/n)n^{-k},$$
$$|f(z)-\pi_n(z, 1/z)|\leq A_4\omega_0(1/n)n^{-k},$$

where z is not outside, or not inside C, or is any point on C, respectively. Theorems 8–10 deal with a converse of this (cf. Walsh II, Theorem 2). Theorem 11 deals with approximation to $f(z)$ (z on C) by functions analytic in an annular region containing C. *H. Kober* (Birmingham).

Hiong, King-Lai. Un théorème général relatif à la croissance des fonctions holomorphes et privées de zéros dans le cercle unité et un nouveau critère de normalité pour une famille de fonctions holomorphes ou méromorphes. C. R. Acad. Sci. Paris **236**, 1322–1324 (1953).

Throughout this review $F(z)$, $f(z)$ stand for functions regular in $|z|=r<1$. For functions $F(z)$ not assuming the values 0 and 1 Schottky's theorem gives an upper bound for $|F(z)|$ in terms of $F(0)$ and r. The author considers functions $f(z)$ which have 0 as exceptional value, and which moreover are of finite order $\leq\lambda$ in $r<1$ (that is, $\log|f(z)|<A(\epsilon)(1-r)^{-\lambda-\epsilon}$ for every $\epsilon>0$). For non-constant functions $f(z)$ for which $f(0)\neq 1$ the author announces an upper bound for $|f(z)|$ in terms of $f(0)$, r and λ. Several consequences of this inequality are stated, the most interesting being the following. Any family of functions $f(z)$ of finite order $\leq\lambda$ satisfying $f(z)\neq 0$ in $r<1$, $f(0)\neq 1$, is normal in $r<1$. The author finally announces extensions of his inequality and some of the consequences to functions of infinite order in $r<1$. *J. Korevaar* (Madison, Wis.).

MacLane, G. R. Sequences of derivatives and normal families. J. Analyse Math. **2**, 72–87 (1952). (Hebrew summary)

Conditions on $f(z)$ for the sequence $f^{(n)}(z)$ to be a normal family without the limit infinity are obtained. Examples of functions $f(z)$ for which the sequence is normal and does admit infinity as limit are given. An integral function $f(z)$ is constructed such that a subsequence of $f^{(n)}(z)$ exists converging in a given simply connected region to any given function regular there. *A. J. Macintyre* (Aberdeen).

Dugué, Daniel. Vers un théorème de Picard global. Ann. Sci. Ecole Norm. Sup. (3) **69**, 65–81 (1952).

This paper is an extension and generalisation of earlier papers [C. R. Acad. Sci. Paris **232**, 380–381, 841–842 (1951); these Rev. **12**, 601; **13**, 452]. It concerns functions possessing

several essential singularities. Such functions were first mentioned by Mittag-Leffler and Poincaré. Maillet [J. Math. Pures Appl. (5) 8, 329–386 (1902)] has dealt in detail with functions having a finite number of isolated essential singularities which he called quasi-integral functions and for which the value infinity is exceptional. He showed that Picard's theorem holds "locally" in the neighbourhood of each singularity, isolated or not, each essential point having at most two exceptional values. It would thus appear that "in the large" there cannot be more than $2n$ exceptional values if there are n essential singularities. The author proves that, in fact, there are at most $n+1$ exceptional values and that this bound is attained for $n=2$ and $n=3$. The results are established for values exceptional-B as well as exceptional-P, but no generalisation of the Nevanlinna theory of defective values is attempted. The paper ends with an interesting list of related problems at present unsolved. *R. Wilson* (Swansea).

Kuramochi, Zenjiro. A remark on the bounded analytic function. Osaka Math. J. 4, 185–190 (1952).

Generalizing a method developed by the reviewer for the case of plane domains [Amer. J. Math. 73, 78–106 (1951); these Rev. 12, 491], the author discusses extremal problems for bounded analytic functions on open Riemann surfaces of positive genus. The procedure is illustrated by a detailed treatment of the case corresponding to the classical Schwarz lemma which had previously been discussed, by different methods, by L. V. Ahlfors [Comment. Math. Helv. 24, 100–134 (1950); these Rev. 12, 90; 13, 1138]. A complete characterization of the extremal function is obtained and Ahlfors' positive differential is constructed. *Z. Nehari.*

Tsuji, Masatsugu. Myrberg's approximation theorem on Fuchsian groups. J. Math. Soc. Japan 4, 310–312 (1952).

Let G be a Fuchsian group of the first kind, having a finite basis, of linear transformations of the circle $|z|\leqq 1$ onto itself. A simple new proof is given of the following special case of a theorem of Myrberg's [Acta Math. 57, 389–409 (1931)]: Under the above condition, there exists a set E of measure 2π on $|z|=1$ with the property that if $e^{i\theta}\in E$, a suitable subsequence of the transforms of the diameter of $|z|=1$ through $e^{i\theta}$ approaches any prescribed circle orthogonal to $|z|=1$. The proof of this result is based on E. Hopf's ergodic theorem [Trans. Amer. Math. Soc. 39, 299–314 (1936)]. *W. Seidel* (Rochester, N. Y.).

Epheser, Helmut, und Stallmann, Friedemann. Konforme Abbildung eines Parallelstreifens mit Halbkreiskerbe. Arch. Math. 3, 276–281 (1952).

The authors consider the conformal mapping of the region obtained by removing the half-circle $|z|<\rho$ $(0<\rho<1)$, $\operatorname{Im}\{z\}>0$, from the strip $0<\operatorname{Im}\{z\}<1$. They show that the rather complicated exact mapping function can be replaced with little loss of accuracy by a simple expression involving logarithms. *Z. Nehari* (St. Louis, Mo.)

***Broman, Arne. Conformal mapping and convergence on the boundary.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 153–157. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let D be a finite plane domain of finite connectivity and let S be the universal covering surface of D. The author studies the convergence, for $|z|=1$, of the series $f(z)=\sum_0^\infty c_n z^n$, where $w=f(z)$ maps $|z|<1$ onto S. A criterion is given. *Z. Nehari* (St. Louis, Mo.).

Lelong-Ferrand, Jacqueline. Sur la représentation conforme des bandes. J. Analyse Math. 2, 51–71 (1952). (Hebrew summary)

Let D represent the strip $|y|<a/2$ in the $z=(x+iy)$-plane. The conformal map $w=f(z)$ is to take D onto a simply connected domain Δ of the $w=(u+iv)$-plane with the boundary point $z=+\infty$ corresponding to the accessible boundary point α at $w=\infty$ having as an arc of access the curve $v=g(u)$, g continuous for all $u\geqq u_0$. Then $f_t(z)=f(z+t)-f(t)$ maps D onto a domain Δ_t obtained from Δ by a translation through $-f(t)$. The author states conditions on Δ in order that $\lim_{t\to\infty} f_t(z)=Cz$, $C=\lambda e^{i\mu}$, and the family of domains Δ_t converge to a strip $|u\cos\mu-v\sin\mu|<a\lambda/2$. For each $u\geqq u_0$, there is a unique vertical segment Θ_u on $\operatorname{Re}(w)=u$ which separates $u_0+ig(u_0)$ from α. Let $\phi(w)$ be the inverse of $f(z)$ and denote by $\bar{X}(u)$ and $\underline{X}(u)$ the sup and inf resp. of $\operatorname{Re}(\phi(w))$ for w on Θ_u. Giving Δ by $u>u_0$, $g_1(u)<v<g_2(u)$, she derives asymptotic estimates for $\phi(w+w_n)-\phi(w_n)$, $w_n=u_n+\frac{1}{2}i(g_1(u_n)+g_2(u_n))$ as $u_n\to\infty$ in terms of $[g_2(u_n)-g_1(u_n)]^{-1}$. She also obtains lower and upper bounds on $\bar{X}(u'')-\underline{X}(u')$ in terms of

$$\int_{u'}^{u''}\frac{du}{g_2(u)-g_1(u)}.$$

G. Springer (Evanston, Ill.).

Krein, S. G. On invariant points in conformal mapping. Uspehi Matem. Nauk (N.S.) 8, no. 1(53), 155–159 (1953). (Russian)

G. N. Položiĭ has shown [Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 203–205 (1952); these Rev. 14, 549] that if a simply-connected region G is mapped conformally onto a simply-connected subregion G_1 which has a simple arc γ of its boundary in common with that of G, then there can be at most three fixed points on γ in the correspondence of the boundaries. In the case that there are exactly three such fixed points, the outer two are attractive while the inner one is repellent. In the case of two fixed points, one is repellent, the other attractive, while a single fixed point is repellent. The author extends Položiĭ's result to the case that the boundaries of G and G_1 have n such arcs in common. It is shown that in each of these arcs, with the possible exception of one of them, there can be at most one fixed point, which is always repellent, while in the exceptional arc there can be at most three fixed points, which follow Položiĭ's rule of attraction. If, however, there is an interior point of G_1 which goes into itself under the mapping, there can be no exceptional arc. *A. J. Lohwater.*

Nehari, Zeev. On weighted kernels. J. Analyse Math. 2, 126–149 (1952).

The author considers kernel functions corresponding to various norms with weight functions. He shows how to calculate the kernel for one weight function in terms of the kernel for another by means of a successive approximation procedure. Estimates of the èrror for the usual orthonormal expansion of the Szegö kernel function are also given. *P. R. Garabedian* (Stanford, Calif.).

Lokki, Olli. Über das Randwertproblem der analytischen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 144, 8 pp. (1952).

For a multiply-connected plane region, the paper gives a construction, based on the Szegö kernel function corresponding to an appropriate weight function, of an analytic function with prescribed modulus on the boundary. *P. R. Garabedian* (Stanford, Calif.).

Lehto, Olli. On the distortion of conformal mappings with bounded boundary rotation. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **124**, 14 pp. (1952).

The author derives a number of basic distortion theorems for the class of analytic mappings of the unit circle which carry the circumference into a curve whose tangent angle has a bounded total variation. *P. R. Garabedian.*

***Schaeffer, A. C., and Spencer, D. C. A variational method for simply connected domains.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 189–191. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The function $w=f(z)=z+a_2z^2+\cdots$ maps $|z|<1$ schlicht onto a simply connected domain D. The authors discuss certain variational methods, previously studied by them, which transform $f(z)$ into a similar function $w^*=f(z)$ that maps $|z|<1$ on a slightly deformed domain D^*. Either an analytical boundary arc of D is deformed; or the unit circle $|z|<1$ itself is slightly changed by deforming an arc in it and thus opening a hole or making it covered twice. Explicit formulae for such deformations are given. Löwner's well-known differential equation for an one-parametric continuous family of mapping functions $f(z, t)$ is obtained as a particular case. *W. W. Rogosinski.*

***Schiffer, M., and Spencer, D. C. Some remarks on variational methods applicable to multiply connected domains.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 193–198. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The authors outline a method for determining the variation of a function $f(z)$ that is regular and schlicht in a multiply connected domain $\mathfrak{M}$ and whose values lie schlicht on another multiply connected domain $\mathfrak{N}$. The variation is effected by a slight deformation of a small analytical Jordan arc lying in the interior of $\mathfrak{M}$. Certain invariant differentials of the domains are involved in the final formulae for the variation of f. *W. W. Rogosinski.*

***Weinstein, Alexander. On the Helmholtz problem of conformal representation.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 105–115. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

L'auteur rappelle les divers résultats déjà obtenus dans la théorie des jets plans et symétriques. Le cas des jets issus d'un conduit polygonal est plus particulièrement étudié, et il est donné une démonstration du théorème d'existence qui repose sur une méthode de continuité, qui est nouvelle par certains côtés. *R. Gerber* (Toulon).

***Bers, Lipman. Some generalizations of conformal mapping suggested by gas dynamics.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 117–124. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

L'auteur envisage les écoulements plans, irrotationnels, subsoniques, d'un gaz compressible pour lequel la densité ne dépend que de la température. Les équations indéfinies de ces mouvements présentent une grande parenté avec les équations de l'hydrodynamique plane, et dans ce travail sont étudiées les extensions possibles des propriétés mathématiques de ces dernières équations à celles des mouvements considérés. Cela conduit l'auteur à définir des "fonctions analytiques généralisées", dont il étudie le développement et l'allure des singularités. Ces fonctions déterminent des transformations planes qui jouissent des propriétés topologiques de la représentation conforme. Ces transformations sont étudiées du point de vue analytique et elles sont utilisées dans la seconde partie du travail à la résolution du problème de l'écoulement subsonique autour d'un profil d'aile. *R. Gerber* (Toulon).

***Steiner, Antonio. Eine direkte Konstruktion der Abelschen Integrale erster Gattung.** Dissertation vorgelegt der Philosophischen Fakultät II der Universität Zürich, 1950. 29 pp.

This dissertation treats the existence of abelian integrals of the first kind using the alternating procedure. H. A. Schwarz, in attempting to establish the existence theorem in question by such methods, encountered difficulties which he was unable to overcome [Ges. Math. Abh., vol. 2, Springer, Berlin, 1890, pp. 303–306]. The difficulties are surmounted by a modification of the alternating procedure of Schwarz which guarantees convergence. The present work was inspired by a related paper of Nevanlinna [Comment. Math. Helv. **22**, 302–316 (1949); these Rev. **10**, 525]. *M. Heins* (Providence, R. I.).

Virtanen, K. I. Über Extremalfunktionen auf offenen Riemannschen Flächen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **141**, 7 pp. (1952).

Let F be the class of Abelian integrals f with a finite Dirichlet integral over a given Riemann surface G. The author proves that the maximum among all functions in F of the derivative of f is either everywhere positive on G, or vanishes identically, while examples show that the same is not true for other classes of bounded analytic functions. *P. R. Garabedian* (Stanford, Calif.).

***Myrberg, P. J. Über automorphe Funktionen und Riemannsche Flächen.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 24–34. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

An account of some recent developments in the theories of automorphic functions and Riemann surfaces. Among the topics considered are: generalizations of the sigma function; relations between the structure of Fuchsoid groups and the existence of functions of specified type on the associated Riemann surface; contrasts between the function theory on Riemann surfaces of finite genus and that on Riemann surfaces of infinite genus. *M. Heins* (Providence, R. I.).

Tamura, Jirô. A note on Riemann surfaces and analytic functions. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **2**, 125–128 (1952).

The author gives a new proof of the fact that, for every Riemann surface, there exists an associated analytic function [Weyl, Die Idee der Riemannschen Fläche, Teubner, Leipzig-Berlin, 1913]. *L. Sario* (Cambridge, Mass.).

Nevanlinna, Rolf. Über die Polygondarstellung einer Riemannschen Fläche. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **122**, 9 pp. (1952).

Let L_z^1, L_z^2 be two disjoint open analytic Jordan arcs issuing from a common point P, in a given one-to-one

analytic correspondence. Cut the angular region between these arcs by a curve L_s^0 connecting two corresponding points on them. By identification of all corresponding points of L_s^1, L_s^2, a Riemann surface R_s is created, bounded by L_s^0 and an ideal boundary component arising from P. The author derives a sufficient condition, similar to the type criterion of Ahlfors [C. R. Acad. Sci. Paris **201**, 30–32 (1935)], for the modulus of R_s to be infinite. Some applications are given. *L. Sario* (Cambridge, Mass.).

Myrberg, Lauri. Über das Verhalten der Greenschen Funktionen in der Nähe des idealen Randes einer Riemannschen Fläche. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **139**, 8 pp. (1952).

The author remarks that the greatest lower bound of the Green's function in a neighborhood of a boundary component is either zero or positive. This property is locally and conformally invariant. *L. Sario* (Cambridge, Mass.).

Taylor, Howard E. Determination of the type and properties of the mapping function of a class of doubly-connected Riemann surfaces. Proc. Amer. Math. Soc. **4**, 52–68 (1953).

The author is concerned with the class of Riemann surfaces constructed essentially as follows. Take an infinite sequence S_j $(j=0, \pm1, \pm2, \cdots)$ of extended planes cut along two slits α_j, β_j such that for each $k=0, \pm1, \pm2, \cdots$ the slits α_{2k} and β_{2k+1} are coincident with α_{2k+1} and β_{2k+2}, respectively. Join S_{2k} to S_{2k+1} identifying crosswise the edges of α_{2k}, α_{2k+1}, and join S_{2k+1} to S_{2k+2} similarly along β_{2k+1}, β_{2k+2}, so as to create a doubly connected Riemann surface F. The author shows that F is of parabolic type and gives an analytic expression for the function mapping the finite plane, punctured at $z=0$, onto F. The reasoning is based on G. R. MacLane's approximation methods [Trans. Amer. Math. Soc. **62**, 99–113 (1947); these Rev. **9**, 85]. *L. Sario* (Cambridge, Mass.).

Volkovyskiĭ, L. I. On the type problem of a simply connected Riemann surface. Ukrain. Mat. Žurnal **1**, no. 1, 39–48 (1949). (Russian)

The author considers Riemann surfaces formed by pasting together infinitely many rectilinear strips according to certain specifications which enable him to apply Ahlfors' criterion, so as to obtain a sufficient condition that these surfaces be of parabolic type. By the method of quasi-conformal mapping, sufficient conditions for the parabolic type of certain classes of Riemann surfaces are deduced from this. *W. Seidel* (Rochester, N. Y.).

Volkovyskiĭ, L. I. An example of a simply connected Riemann surface of hyperbolic type. Ukrain. Mat. Žurnal **1**, no. 3, 60–67 (1949). (Russian)

An example is constructed of a simply connected Riemann surface of hyperbolic type which has two nonalgebraic branch points over the points $w=0$ and $w=1$ and infinitely many algebraic branch points over the point $w=\infty$. [It should be noted that G. Valiron, J. Math. Pures Appl. (9) **15**, 423–435 (1936), has given examples of simply connected Riemann surfaces of hyperbolic type which have a single nonalgebraic branch point isolated from the algebraic branch points.] *W. Seidel* (Rochester, N. Y.).

Niini, Risto. Über eine nichtkonstruierbare Riemannsche Fläche vom Geschlecht Eins. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **132**, 6 pp. (1952).

The existence of a Riemann surface W with the following topological properties is studied: (1) W is a closed covering surface of the Riemann sphere; (2) the genus of W is one; (3) the orders of branch points are 7, 3, 2; (4) no sheet has several branch points of the same order; (5) the number of sheets is 21, the smallest compatible with the Riemann-Hurwitz-formula. The author proves that there do not exist Riemann surfaces with these properties; surfaces with 28 sheets, the next compatible number, can easily be constructed. *L. Sario* (Cambridge, Mass.).

Theory of Series

Kruskal, Joseph B., Jr. Monotonic subsequences. Proc. Amer. Math. Soc. **4**, 264–274 (1953).

The starting point of the investigations of the author is the following theorem of Szekeres and the reviewer [Compositio Math. **2**, 463–470 (1935), p. 467]: Every sequence of n^2+1 real numbers possesses a (perhaps not strictly) monotonic subsequence having $n+1$ terms. n^2+1 is the smallest number for which this is true.

Let $\{X_i\}$ be a sequence of vectors from a finite-dimensional real vector space. The author defines the sequence to be monotonic if the vectors $X_{i+1}-X_i$ all lie in a closed half-space determined by a hyperplane through the origin. Let $\psi_r(n)$ be the smallest number of vectors in r-space which necessarily contains a monotonic subsequence of n terms. The theorem of Szekeres and the reviewer states that $\psi_1(n+1)=n^2+1$. The author conjectures that

$$\psi_r(n+r)=rn+n^2-n+1.$$

He proves that $\psi_r(2+r)\leqq 2r+3$, with equality for $r=1$ and 2. The proof is not easy. Several other generalizations are discussed. *P. Erdös* (Los Angeles, Calif.).

Zeller, Karl. Sur la méthode de sommation d'Abel. C. R. Acad. Sci. Paris **236**, 568–569 (1953).

Au moyen d'une suite de semi-normes, une topologie peut être introduit dans l'espace A^* de toutes les séries $\sum a_k$ sommables par le méthode d'Abel A. Comme conséquences, on obtient que la méthode A est parfaite et qu'il n'existe aucune méthode à lignes finies équivalente à A. La forme générale des facteurs f_k tels que $\{a_k\}\,\epsilon A^*$ entraîne $\{f_k a_k\}\,\epsilon A^*$ est donné par la formule $f_k=\int_0^1 t^k d\chi(t)+O(\sigma^k)$, $0<\sigma<1$, $\chi(t)$ étant une fonction à variation bornée. *G. G. Lorentz* (Detroit, Mich.).

Jackson, Frederick H. Inclusion theorems for summability matrices of variable dilution. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 52–62 (1953).

The "méthode de sommabilité par des moyennes éloignées" of Mollerup [Danske Vid. Selsk. Math.-Fys. Medd. **3**, no. 8 (1920)] and the "deferred Cesàro means" of Agnew [Ann. of Math. (2) **33**, 413–421 (1932)] can be derived from the Cesàro transformation of order one by appropriate dilution of the $(C, 1)$ matrix with zeros. The author adds to his recent work on diluted matrices [Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 173–180, 181–190

(1952); these Rev. 13, 933] by investigating deferred Cesàro means of order r. In particular, he finds certain conditions under which the convergence field of a diluted (C, r) matrix contains the convergence field of the undiluted (C, r) matrix as a proper subset. *G. Piranian* (Ann Arbor, Mich.).

Vermes, P. Convolution of summability methods. J. Analyse Math. 2, 160–177 (1952).

The author defines a convolution $C=A*B$ of two sequence-to-sequence methods $A=(a_{nk})$, $B=(b_{nk})$ by

$$c_{nk}=a_{nk}b_{n0}+a_{n,\,k-1}b_{n1}+\cdots+a_{n0}b_{nk}.$$

[This is different from the definition usual for Nörlund methods, where $(N, p_k)*(N, q_k)$ is the method (N, r_k) with $r_n=p_nq_0+\cdots+p_0q_n$; for instance, the convolution of convergence with itself in the author's sense is the method $s_{2n}\to s$.] The convolution E_t*E_s of two Euler-Knopp methods may sum the series $\sum z^n$ in two disjoint domains. The author gives also a variant of the Borel-Okada theorem on summability of power series where the set of summability of $\sum z^n$ is not connected. Finally, he discusses the convolution $H=G**F$ of two series-to-sequence methods $F=(f_{nk})$, $G=(g_{nk})$ defined by $h_{nk}=(f_{nk}g_{n0}+\cdots+f_{n0}g_{nk})/(k+1)$; this is essentially different from his first definition even for simplest methods. Several of the author's quotations are not exact, for instance, the theorem under (3.6) follows in a trivial way from one by Agnew [Amer. J. Math. 61, 178–186 (1939)]. *G. G. Lorentz* (Detroit, Mich.).

Maschler, Michaël. Prolongement analytique par la méthode de la transformation généralisée de série en série. C. R. Acad. Sci. Paris 236, 883–885 (1953).

The author exhibits formulas for certain series-to-series transformations. These transformations represent analytic continuations obtained by the composition of n direct analytic continuations. The case where $n=1$ has recently been treated by Meyer-König [Math. Z. 52, 257–304 (1949); these Rev. 11, 242], Vermes [Amer. J. Math. 72, 615–620 (1950); these Rev. 12, 20], Cowling [Proc. Amer. Math. Soc. 1, 536–542 (1950); these Rev. 12, 91], and others. *G. Piranian* (Ann Arbor, Mich.).

Rodeja F., E. G.-. Note on the sum of powers of sines, cosines and tangents. Euclides, Madrid 13, 20–24, 115–119 (1953). (Spanish)

Spiegel, M. R. Some interesting series resulting from a certain Maclaurin expansion. Amer. Math. Monthly 60, 243–247 (1953).

Tanaka, Chuji. Note on Dirichlet series. IV. On the singularities of Dirichlet series. Proc. Amer. Math. Soc. 4, 308–309 (1953).

Given a Dirichlet series $\sum a_ne^{-\lambda_ns}$ with $\lim(\log n)\lambda_n^{-1}=0$ and σ_s as its abscissa of simple convergence, the author proves the existence of a series $\sum b_ne^{-\lambda_ns}$ having the line $\sigma=\sigma_s$ as natural boundary and such that either $|a_n|=|b_n|$ and $\arg a_n-\arg b_n\to0$ or $\arg a_n=\arg b_n$ and $b_n/a_n\to1$. For the construction the author chooses a sequence of integers $\{n_i\}$ such that $\sigma_s=\lim\lambda_{n_i}^{-1}\log|a_{n_i}|$ and $\liminf(\lambda_{n_{i+1}}-\lambda_{n_i})>0$. He then sets $b_n=a_n$ for $n\neq n_i$ and $b_n=a_n\exp(\alpha\theta/\lambda_n)$ for $n=n_i$. Here $\alpha=(-1)^{1/2}$ in the first case and -1 in the second and a cardinality argument is used to show the existence of real numbers θ for which the resulting series has a natural boundary. *E. Hille* (New Haven, Conn.).

Fourier Series and Generalizations, Integral Transforms

Men'šov, D. E. Certain questions from the theory of trigonometric series. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1950, no. 8, 3–10 (1950). (Russian)

An expository lecture reviewing, in detail, certain problems and achievements of the theory of convergence and divergence of trigonometric series, and of Fourier series in particular. *A. Zygmund* (Chicago, Ill.).

Agarwal, S. S. A theorem for the convergence of the conjugate series of a Fourier series. Proc. Nat. Acad. Sci. India. Sect. A. 15, 100–105 (1946).

Suppose $f(x)$ is integrable on the interval $[-\pi, \pi]$ and that $g(x)=(2\pi)^{-1}\int_0^\pi\psi(t)\cot\frac{1}{2}tdt$, where

$$\psi(t)=f(x+t)-f(x-t),$$

is its conjugate function. It is shown that the series conjugate to the Fourier series of $f(x)$ converges to $g(x)$ at every point x where $g(x)$ exists, provided

$$\lim_{k\to\infty}\limsup_{\epsilon\to0}\int_{k\epsilon}^{\pi}\left|\frac{|\psi(t+\epsilon)|}{t+\epsilon}-\frac{|\psi(t)|}{t}\right|dt=0,$$

a criterion which is shown to include Young's test. *G. Klein* (South Hadley, Mass.).

Tandori, Károly. Bemerkung zur Divergenz der Fourierreihen stetiger Funktionen. Publ. Math. Debrecen 2, 191–193 (1952).

Using the trigonometric polynomials which were introduced by Fejér to construct continuous functions with divergent Fourier series, the author defines a continuous function whose Fourier series has uniformly bounded partial sums but nevertheless diverges on a set of points whose intersection with every interval has the power of the continuum. *W. Rudin* (Rochester, N. Y.).

Edrei, A., and Szegö, G. A note on the reciprocal of a Fourier series. Proc. Amer. Math. Soc. 4, 323–329 (1953).

Let $f(\theta)$ and $[f(\theta)]^{-1}$ be non-negative integrable functions in $(0, 2\pi)$ and their Fourier series be $\sum a_\nu e^{i\nu\theta}$ and $\sum b_\nu e^{i\nu\theta}$, respectively. The authors prove the formula:

$$b_{k-j}=\lim_{n\to\infty}(-1)^{j-k}\frac{A_{2n+1}(j,k)}{A_{2n+1}},$$

where

$$A_{2n-1}=\begin{vmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-2n}\\ a_1 & a_0 & a_{-1} & \cdots & a_{-2n+1}\\ a_2 & a_1 & a_0 & \cdots & a_{-2n+2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ a_{2n} & a_{2n-1} & a_{2n-2} & \cdots & a_0 \end{vmatrix}$$

and $A_{2n+1}(j,k)$ is the minor obtained by deleting the $(n+1+j)$th row and the $(n+1+k)$th column of A_{2n+1}. The inequality $A_{2n+1}>0$ is known from G. Szegö, Math. Z. 6, 167–202 (1920). *S. Izumi* (Tokyo).

Singh, U. N. On the strong summability of the derived series of a Fourier series. Proc. Nat. Acad. Sci. India. Sect. A. 15, 63–72 (1946).

A series whose partial sums are S_n is said to be summable H_k $(k>0)$ to S if $\sum_{k=1}^n|S_n-S|^k=o(n)$. For $k=1$ this is strong $(C, 1)$ summability. An immediate consequence of the fact that a Fourier series (as well as its conjugate) is summable H_2 [Marcinkiewicz, J. London Math. Soc. 14,

162–168 (1939); these Rev. 1, 11] is that the same may be said of the derived Fourier series of an absolutely continuous function. The example $f(x)=(\pi-x)/2$ for $0<x<2\pi$ and $f(0)=0$ is shown to have a derived Fourier series not summable H_2 anywhere, thus proving that absolute continuity cannot be replaced by bounded variation in the result above. *G. Klein* (South Hadley, Mass.).

Walmsley, C. Gibbs phenomena for Cesàro and Hölder summation of generalized Fourier series. J. London Math. Soc. 28, 148–156 (1953).

Fourier series are considered for the functions $\cot \frac{1}{2}x$ and $\csc^2 \frac{1}{2}x$, where the coefficients are calculated, respectively, as a Cauchy principal value and as an integral over an arc other than the real axis. Gibbs phenomena are shown to exist for the $(C, 1)$ means in the former case and for the $(C, 2)$ and $(H, 2)$ means in the latter. *P. Civin.*

Rudin, Walter. A remark concerning Graves' closure criterion. Canadian J. Math. 5, 194–195 (1953).

D. P. Dalzell [J. London Math. Soc. 20, 87–93, 213–218 (1945); these Rev. 7, 437] has developed a method to prove closure or completeness of orthonormal sets by adroit application of Parseval's theorem. This method was recently summarized as a closure criterion by R. E. Graves [Canadian J. Math. 4, 198–203 (1952); these Rev. 13, 936]. The present author points out that Graves' conditions for the applicability of the criterion may be slightly weakened.

J. Korevaar (Madison, Wis.).

Berezanskiĭ, Yu. M. On generalized almost periodic functions and sequences connected with differential and difference equations. Mat. Sbornik N.S. 32(74), 157–194 (1953). (Russian)

Let $\omega(t, \lambda)$, $0\leq t<\infty$, denote a solution of the differential equation $y''-q(t)y=-\lambda y$ where $q(t)$ is non-negative, continuous, even, and $o(t^{-2-\epsilon})$, $\epsilon>0$. The initial conditions are $\omega(0, \lambda)=1$, $\omega_t'(0, \lambda)=0$. If $x(t)$ is an even function with continuous second order derivatives, we consider the operator $(T_s x)(t)=u(t, s)$ where $u(t, s)$ is the solution of the hyperbolic equation $\partial^2u/\partial t^2-q(t)u=\partial^2u/\partial s^2-q(s)u$ which corresponds to the initial conditions $u(t, 0)=x(t)$, $u_s'(t, 0)=0$. The operator T_s is called a generalized translation and it has the properties $(T_s x)(t)=(T_t x)(s)$, $T_s T_{s_1}x=T_{s_1}T_s x$, besides continuity and boundedness. If the family of translated functions $f_s(t)=(T_s f)(t)$ is compact with respect to uniform convergence, the function $f(t)$ is called almost periodic with respect to T_s. If we introduce a mean value by Banach's method, we have the following Parseval equation

$$M[|f(t)|^2]=\sum_\lambda |M_t f[(t)\omega(t, \lambda)]|^2\frac{1}{\rho^2(\lambda)}$$

where $\rho^2(\lambda)=M_t[\omega(t, \lambda)^2]$ and the summation is extended at most over a denumerable set of values of λ. To a given $\epsilon>0$ corresponds a translation T_s and numbers $\lambda_1, \cdots, \lambda_n$, $c_1, \cdots, c_n$ such that $|(T_s f)(t)-\sum_{\nu=1}^n c_\nu\omega(t, \lambda_\nu)|<\epsilon$, $0\leq t<\infty$. The function $f(t)$ is called almost periodic in the stronger sense if there always exists a relatively dense set of numbers τ satisfying $|(T_{s+\tau}f)(t)-(T_s f)(t)|<\epsilon$ for all s and t. For this class of functions we have the strong approximation theorem: to $\epsilon>0$ corresponds a finite sum $f_n(t)=\sum_{\nu=1}^n c_\nu\omega(t, \lambda_\nu)$ such that $|f(t)-f_n(t)|\leq\epsilon$ for all values of t.

The polynomials $P_n^{(\alpha,\beta,h)}(\lambda)$, orthonormal with respect to the measure

$d\tau(\lambda)=h(\lambda)(1-\lambda)^\alpha(1+\lambda)^\beta d\lambda$; $\alpha\geq-\frac{1}{2}$, $\beta\geq-\frac{1}{2}$; $-1\leq\lambda\leq1$, where $h(\lambda)$ is >0 and satisfies a Lipschitz condition, are called almost Jacobian. A sequence $x_0, x_1, \cdots$ of complex numbers is transformed by a generalized translation operator

$$(T_k x)_j=\sum_{r=0}^\infty \int_{-1}^1 P_j^{(\alpha,\beta,h)}(\lambda)P_k^{(\alpha,\beta,h)}(\lambda)P_r^{(\alpha,\beta,h)}(\lambda)d\tau(\lambda)x_r.$$

The sequence $f_0, f_1, \cdots$ is called almost periodic if the set of translated sequences $(T_k f)_j$ is compact with respect to uniform convergence. If we define a mean value by Banach's method and put $\rho^2(\lambda; \alpha, \beta, h)=M_j[|P_j^{(\alpha,\beta,h)}(\lambda)|^2]$ and assume that $M_k[M_j[|(T_k T_k f)_j|^2]]<\infty$, we have the Parseval equation

$$M_j[|f_j|^2]=\sum_\lambda |M_j[f_j P_j^{(\alpha,\beta,h)}(\lambda)]|^2\frac{1}{\rho^2(\lambda; \alpha, \beta, h)}.$$

To $\epsilon>0$ corresponds a translated sequence $(T_k f)_j$ which can be approximated uniformly with the accuracy ϵ by a finite sum $\sum_{\nu=1}^n c_\nu P_j^{(\alpha,\beta,h)}(\lambda_\nu)$. If the sequence is almost periodic in the stronger sense, i.e., if there exists a relatively dense set of numbers r, for which $|(T_{k+r}f)_j-(T_k f)_j|<\epsilon$ for all k and j, the approximation holds for the sequence itself. Finally, the author considers the special case $\alpha=\beta=-\frac{1}{2}$ (almost Čebyšev polynomials) and in this case he proves that the system of almost periodic sequences is the exact closure of the system of finite sums. *H. Tornehave* (Copenhagen).

Singh, Udita Narayana. Sur quelques théorèmes de Hille et Tamarkin. C. R. Acad. Sci. Paris 236, 885–887 (1953).

With every complex function f, subject to suitable integrability conditions on $(-\infty, \infty)$, one can associate two functions f_1, f_2, regular in the upper and lower half-planes, respectively, which are defined by the integral $(1/2\pi i)\int_{-\infty}^\infty f(t)(t-z)^{-1}dt$. The Fourier transform of f may also be regarded as a pair of functions g_1, g_2, regular in the upper and lower half-planes. The author states three theorems about the relations between the vanishing of the Fourier transform on a half line, and the boundary behavior of the above analytic functions, for $f \in L_p$ $(1\leq p<\infty)$. There are no proofs. *W. Rudin* (Rochester, N. Y.).

***Petersen, Richard. Laplace-transformation of almost periodic functions.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 158–165. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Démonstration nouvelle d'un résultat de l'auteur. Posant

$$F(s)=\int_0^{+\infty} e^{-st}f(t)dt \quad (\Re(s)>0),$$

où $f(t)$ est une fonction presque périodique de la variable réelle t, d'exposants $(\Lambda_1, \Lambda_2, \cdots)$, $F(s)$ peut être prolongée dans toute domaine ne contenant pas la fermeture de $(i\Lambda_1, i\Lambda_2, \cdots)$; si Λ_k est isolé, $F(s)$ a un pôle simple en ce point. Une autre expression de $F(s)$ permet de montrer que si les exposants de $f(t)$ sont bornés en module, il existe une fonction entière $g(s)$ telle que: $g(it)=f(t)$. *J. Favard.*

Helwig, W. F. Boundary conditions in the Fourier integral formulation. Texas J. Sci. 5, 102–105 (1953).

Some remarks are made concerning the elementary application of Laplace transforms to simple circuit problems.

R. V. Churchill (Ann Arbor, Mich.).

Sumner, D. B. A convolution transform admitting an inversion formula of integro-differential type. Canadian J. Math. **5**, 114–117 (1953).

The author exhibits a simple convolution transform: $f(x)=\int_{-\infty}^{\infty}G(x-t)\phi(t)dt$, where the inversion operator is of the integro-differential type and not a purely differential operator.
S. Agmon (Jerusalem).

Horváth, J. Sur les fonctions conjuguées à plusieurs variables. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 17–29 (1953).

The Poisson formula and the theory of the Hilbert transform are discussed for functions of n-vectors (which are denoted by bold letters). The formulae are of the form $\int K(\mathbf{t}-\mathbf{x}, y)f(\mathbf{t})d\mathbf{t}$, where

$$\text{(1)}\qquad K(\mathbf{x}, y)=C_n^{-1}y(|\mathbf{x}|^2+y^2)^{-(n+1)/2},\quad C_n=\pi^{(n+1)/2}/\Gamma((n+1)/2)$$

for the Poisson integral, (2) $\mathbf{K}(\mathbf{x}, y)=C_n^{-1}\mathbf{x}/(|\mathbf{x}|^2+y^2)^{(n+1)/2}$ for the conjugate Poisson integral, and

$$\text{(3)}\qquad \mathbf{K}(\mathbf{x}, y)=C_n^{-1}\mathbf{x}/|\mathbf{x}|^{(n+1)/2}\quad\text{if}\quad |\mathbf{x}|>y,$$

and 0 otherwise, for the Hilbert transform. It is shown that if $f(\mathbf{t})/(1+|\mathbf{t}|^n)$ is summable over the whole space, then the limit of the integral in case (1) is almost everywhere $f(\mathbf{t})$, and that under the same conditions the limits of the integrals in cases (2) and (3) are the same, almost everywhere. If the limit in case (3) is denoted by $\mathbf{g}(\mathbf{x})$, it is shown that if $f(\mathbf{t})$ is in L^2 then $\mathbf{g}(\mathbf{x})$ exists almost everywhere and has the same norm in L^2 as $f(\mathbf{t})$, and that

$$-f(\mathbf{t})=\lim_{y\to 0}\int \mathbf{K}(\mathbf{x}-\mathbf{t})\mathbf{g}(\mathbf{x})d\mathbf{x}$$

almost everywhere; and the last result is proved also for f in L^p $(1<p<\infty)$.
J. L. B. Cooper (Cardiff).

Humbert, Pierre. A propos des fonctions de Bessel à deux variables. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 19–22 (1953).

The author obtains operational images of

$$\pi J_n(x, y)=\int_0^\pi \cos(nu-x\cos u-y\cos 2u)du$$

for $n=1, 2, 3$, in various forms, and makes some applications.
A. Erdélyi (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Lyusternik, L. A. On polynomial approximation of functions given in the whole plane. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 161–164 (1953). (Russian)

S. Bernstein [Bull. Soc. Math. France **52**, 399–410 (1924)] proved that if $f(x)$ is continuous and $f(x)e^{-|x|}\to 0$ as $|x|\to\infty$, then $f(x)$ admits approximation by polynomials with the weight $e^{-|x|}$, i.e., there is a sequence of polynomials $P(x)$ such that $\sup_x|f(x)-P(x)|e^{-|x|}\to 0$. The author proves, by reducing it to the one-variable case, the corresponding theorem for two variables: if $f(r, \phi)$ is continuous in the whole plane and $\sup_\phi|f(r, \phi)|e^{-r}\to 0$ as $r\to\infty$, then there are polynomials $P(x, y)$ such that

$$\sup_r\{\sup_\phi|f(r, \phi)-P(r\cos\phi, r\sin\phi|\}e^{-r}\to 0.$$

R. P. Boas, Jr. (Evanston, Ill.).

Geronimus, Ya. L. On some estimates for polynomials. Doklady Akad. Nauk SSSR (N.S.) **88**, 193–196 (1953). (Russian)

(1) Let H_p' $(p>1)$ denote the class of regular functions $\phi(z)$ of H_p whose boundary functions $\phi(e^{i\theta})$ belong to Lip $(1/p, p)$. Let $R_n(z)$ be a system of polynomials such that, in the L^p metric, $\|\phi-R_n\|=O(n^{-1/p})$. If $|R_n(e^{i\theta})|$ is bounded, uniformly in n and θ, then $\phi(z)$ is bounded in the unit circle, and conversely; and if

$$M_n=\max_{|z|\le 1}|R_n(z)|=|R_n(e^{i\theta_0})|,\quad M_n\sim|\phi(re^{i\theta_0})|$$

when $1-r=O(1/n)$.

(2) Let $P_n(e^{i\theta})$ be orthonormal with respect to the weight $d\sigma(\theta)$. If (i) $\epsilon_n=\sum_{k=n+1}^{\infty}|P_k(0)|^2=O(1/n)$, $\{P_n(z)\}$ is uniformly bounded on $|z|=1$ if and only if

$$\pi(z)=\exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\sigma'(\theta)d\theta\right\}$$

is bounded in $|z|<1$; and

$$\text{(ii)}\qquad M_n=\max|P_n^*(z)|=|P_n^*(e^{i\theta_0})|\sim\pi(re^{i\theta_0}),\quad 1-r=O(1/n),\quad P_n^*(z)=z^n\bar{P}_n(1/z).$$

The author gave a sufficient condition on $d\sigma(\theta)$ for (i) to be true [same Doklady **83**, 5–8 (1952); these Rev. **13**, 740]. Now he gives one bearing on the leading coefficients α_n of $P_n(z)$: (ii) is true if $\alpha_n-\alpha=O(1/n)$ with a positive α.
R. P. Boas, Jr. (Evanston, Ill.).

Grosswald, Emil. On some algebraic properties of the Bessel polynomials. Trans. Amer. Math. Soc. **71**, 197–210 (1951).

For integer n, the differential equation

$$x^2y''+2(x+1)y'-n(n+1)y=0$$

is satisfied by the polynomials

$$y_n(x)=\sum_{\nu=0}^{n}(n+\nu)!(x/2)^\nu/\{(n-\nu)!\nu!\}.$$

They have been studied by S. Bochner [Math. Z. **29**, 730–736 (1929)] and by others. The present paper deals with the following topics. 1. Asymptotic formulas of $y_n(x)$ for large n are proved. [These results were not completely clear to the reviewer.] 2. The zeros of $y_n(x)$ are shown to be simple and to lie in $|x|<1$, except for $n=1$ when $x=-1$. For even n, there are no real zeros, for odd n there is exactly one real zero. Let x_n be this real zero, n odd. Then x_n is increasing from -1 to 0 as n increases. 3. The polynomials $y_n(x)$ are irreducible for $n\leqq 400$ and also for certain $n>400$. In case they are reducible, they contain an irreducible factor of degree A_nn, where $A_n>16/17$ and $\to 1$ as $n\to\infty$. 4. The Galois group of an irreducible $y_n(x)$ is the symmetric group.
G. Szegö (Stanford, Calif.).

Schmutz, O. Koeffizientenbedingungen zur Kontrolle des Dämpfungsgrades bei Ausgleichvorgängen (verallgemeinerte Hurwitzbedingungen). Ing.-Arch. **21**, 33–41 (1953).

Proceeding from the well-known Hurwitz stability criterion, insuring that the roots of an algebraic equation with real coefficients have negative real parts, the author derives a similar criterion which insures that the roots lie in a prescribed sector containing the negative real axis in the complex plane. Some alternative forms of the criterion are given, and applications to problems of control are indicated.
L. A. MacColl (New York, N. Y.).

Special Functions

***Sibagaki, Wasao. Theory and applications of the gamma function with a table of the gamma function for complex arguments significant to the sixth decimal place.** Iwanami Syoten, Tokyo, 1952. 202 pp. Yen 680. (Japanese)

As far as a person ignorant of Japanese is able to judge this is an excellent and self-contained monograph on the gamma function. Compared with the recent book by Lösch and Schoblik [Die Fakultät (Gammafunktion) . . . , Teubner, Leipzig, 1951; these Rev. 13, 938] the present publication is more restricted in its scope (no incomplete gamma functions or related functions, no applications), gives more material for numerical work, and requires less background.

Chapter I (43 pp.) gives background material, mostly on function theory (elementary functions, their Riemann surfaces, conformal mappings, expansions, altitude charts, relief diagrams, etc.). Chapter II (41 pp.) introduces Euler's and Weierstrass' products and Euler's integral for $\Gamma(z)$. Most of the basic properties of the gamma function are proved in this chapter. The logarithmic derivative $\Psi(z)$ is also introduced as is Bernoulli's polynomial. Chapter III (20 pp.) gives graphs of $\Gamma(x)$ and $\Psi(x)$ for real x, altitude charts for complex variable, and a relief diagram; also formulas and methods for numerical computation. Chapter IV (14 pp.) introduces Euler's beta function and the hypergeometric function. Appendix I (9 pp.) contains theorems on term-by-term integration of infinite series, infinite integrals depending on a parameter, etc. Appendix II (4 pp.) gives Salzer's coefficients for complex interpolation. There are three pages of exercises, 53 pages of numerical tables, and a nine-page formulaire with references to the text.

The principal tables (pp. 135–187) give 6 D values of the real and imaginary parts of $\ln \Gamma(x+iy)$ for

$$x=-10(.2)-6(.1)10.4, \quad y=0(.1)2(.2)10.$$

On p. 18 there is a 16 decimal table of $S(n)=\sum_{k=1}^{\infty}k^{-n}$ for $n=2(1)20$. On p. 90 there are 6 D values of the first ten zeros of $\Psi(z)$, together with 6 D or 5 S corresponding values of $\Gamma(z)$ and $\ln|\Gamma(z)|$. There is a further auxiliary table on pp. 103, 104. *A. Erdélyi* (Pasadena, Calif.).

Buchholz, Herbert. Die Lösungen einer besonderen Whittakerschen inhomogenen Differentialgleichung. Math. Z. 57, 167–192 (1953).

The author studies the solutions of the differential equation

$$y''+[\tfrac{1}{4}(1-\mu^2)z^{-2}+\kappa z^{-1}-\tfrac{1}{4}]y=Ae^{-z/2}z^{(\nu-3)/2}$$

where A is either $\{\Gamma[(\nu-\mu)/2]\Gamma[(\nu+\mu)/2]\}^{-1}$ or 1, the two cases being essentially different when $\nu\pm\mu$ is an even integer. He obtains particular solutions as series of ascending or descending powers of z, an integral representation of the Mellin-Barnes type, investigates the asymptotic behavior of the solution as $z\to\infty$, gives recurrence relations, etc., correlates his solutions with those obtained by variation of parameters (the solutions of the reduced equation being Whittaker's functions), and gives a considerable number of integral representations and integral relations.

The functions studied in this paper are related to Whittaker's function very much as Lommel's functions [G. N. Watson, Theory of Bessel functions, Cambridge, 1944, §§10.7–10.75; these Rev. 6, 64] are related to Bessel functions, and the theory of the new functions is developed far enough to include the known theory of Lommel's functions. When $\kappa=0$, the functions discussed in this paper reduce in essence to Lommel's functions, and for any κ they may be expressed as Laplace integrals of Lommel's functions. [Reviewer's remark: The functions studied in this paper are instances of MacRobert's E-function, and hence also of Meijer's G-function.] *A. Erdélyi* (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. III. Nederl. Akad. Wetensch. Proc. Ser. A. 56 = Indagationes Math. 15, 43–49 (1953).

[For parts I and II see same Proc. 55, 369–379, 483–487 (1952); these Rev. 14, 469, 642.] In the present instalment the author proves his "first expansion theorem"

$$\left\{\prod_{j=1}^{k}\Gamma(1-c_j)\right\}^{-1} G_{p+k,\,q+k}^{m,\,n+k}\left(\lambda w\,\middle|\,\begin{matrix}c_j,\,a_r\\ b_s,\,d_j\end{matrix}\right)$$
$$=\left\{\prod_{j=1}^{k}\Gamma(1-d_j)\right\}^{-1}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,{}_{k+1}F_k(-r,1-c_j;1-d_j;\lambda)$$
$$\times G_{p+1,\,q+1}^{m,\,n+1}\left(w\,\middle|\,\begin{matrix}0,\,a_r\\ b_s,\,r\end{matrix}\right)$$

under several sets of appropriate conditions. For $k=0$, this expansion reduces to the one proved in part I. *A. Erdélyi* (Pasadena, Calif.).

Matsumoto, Toshizô. Note on the integral representation of Mathieu functions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 133–137 (1952).

Verf. setzt sich zum Ziel, die Kerne der linearen homogenen Integralgleichungen für die Mathieuschen Funktionen, welche im Buch von Whittaker und Watson [Modern analysis, 4th ed., Cambridge, 1927], sowie im Buch von McLachlan [Theory and application of Mathieu functions, Oxford, 1947; diese Rev. 9, 31] vorkommen, zu verallgemeinern. Hierzu geht er von der Whittakerschen Integralgleichung für die ungeraden Mathieuschen Funktionen aus. Die betreffende Gleichung gilt für die ungeraden Funktionen ungerader Ordnung, aber nicht für jene gerader Ordnung. Er zeigt, dass für die Kerne der periodischen Mathieuschen Funktionen ein allgemeiner Ausdruck gilt, welcher aus der Differentialgleichung abgeleitet wird, und welcher 2 willkürliche Funktionen enthält. Indem diese willkürlichen Funktionen durch trigonometrische Funktionen ersetzt werden, erhält man die bekannten Kerne, welche oben genannt wurden. Analoge Ueberlegungen werden auf die Differentialgleichung der Laméschen Potentialfunktionen in der Jakobischen Form angewandt, sowie auf eine weitere Differentialgleichung mit trigonometrischen Koeffizienten. Zum Schluss bringt Verf. einige allgemeine Ueberlegungen bezüglich Kerne von Integralgleichungen für periodische Lösungen von linearen homogenen Differentialgleichungen zweiter Ordnung mit periodischen Koeffizienten. *M. J. O. Strutt* (Zürich).

Koster, G. F. Localized functions in molecules and crystals. Physical Rev. (2) 89, 67–77 (1953).

The energy band theory of solids is based on the assumption of the existence of an effective Hamiltonian containing a periodic potential. By taking essentially a Fourier transform of the solutions of the Schrödinger equation, which are extended over the whole crystal, Wannier [Physical Rev. (2) 52, 191–197 (1937)] introduced a new set of functions, which were localized about the atoms of the crystals and which were mutually orthogonal. These functions have

proved to be useful in many problems, and the purpose of the present paper is to give more direct ways of calculating them. The Wannier functions in a crystal are here defined in terms of a differential equation, and the corresponding variation principle is stated for the case of a fixed periodic potential. By means of group theory, the variation procedure is also used to define localized orthogonal functions in molecules, which have many properties of the Wannier functions. Solutions of the Schrödinger equation can be obtained from the localized functions, and examples of this procedure are taken from crystalline and molecular problems. A numerical method of carrying out the variation calculation is also discussed. *P. O. Löwdin* (Uppsala).

Harmonic Functions, Potential Theory

Tsuji, Masatsugu. On F. Riesz' fundamental theorem on subharmonic functions. Tôhoku Math. J. (2) 4, 131–140 (1952).

The proof herein offered of the Riesz decomposition theorem for subharmonic functions is based on the following method of arriving at the mass distribution. Let u be a subharmonic function on a plane domain D, and let $\Delta(r, z)$ denote the open disc $\{\zeta: |\zeta-z|<r\}$ $(\subset D)$. The circumferential mean function $L(r, z; u)=(1/2\pi)\int_0^{2\pi}u(z+re^{i\theta})d\theta$ has the property that $dL(r, z; u)/dr$ exists $(\geqq 0)$ except for at most a countable number of values of r. To each non-exceptional disc there is assigned the mass $\mu[\Delta(r, z)]=rdL(r, z; u)/dr$. Covering any set $e(\subset D)$ by a countable number of non-exceptional discs $\Delta(r_n, z_n)$ and setting

$$\mu^*(e)=\inf \sum_n \mu[\Delta(r_n, z_n)]$$

gives rise to a metric outer measure μ^*, and μ^* confined to the Borel subsets of D is a positive mass distribution. That this mass distribution yields the Riesz decomposition is proved by using convergence properties of convex functions, together with the usual classical approximation techniques. *M. G. Arsove* (Seattle, Wash.).

Kubo, Tadao. On the potential defined in a domain. Proc. Japan Acad. 25, 123–125 (1949).

Let R be a Jordan region, $g(z, \zeta)$ the Green's function for R, and C_0 the level curve $g(z, z_0)=c$ (>0). For V the Green's potential of a mass distribution μ on a Borel subset E of R, the mean $[\int_{C_0}V(z)d\sigma_z]/[\int_{C_0}d\sigma_z]$, where $d\sigma_z$ is the hyperbolic metric, has the following properties: (1) it is equal to $c\mu(E)$ if E lies inside C_0; (2) it is equal to $V(z_0)$ if E lies outside C_0. These results are elementary for the case of R a circle centered at z_0, and the general case reduces to this by a conformal transformation. *M. G. Arsove.*

Weinstein, Alexander. Generalized axially symmetric potential theory. Bull. Amer. Math. Soc. 59, 20–38 (1953).

The author deals with the theory of a special class of linear partial differential equations with variable coefficients which are rational functions of the independent variables. The simplest elliptic case is the Laplace equation. The class under consideration can be derived from the Laplace equation

$$(*)\qquad \frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}+\cdots+\frac{\partial^2 f}{\partial x_n^2}$$

by introduction of a particular type of symmetry regarding axially symmetric solutions of (*) depending on two variables $x=x_1$ and $y=(x_2^2+x_3^2+\cdots+x_n^2)^{\frac{1}{2}}$. Under these restrictions f becomes a function $\Phi(x, y)$ satisfying the equation

$$\frac{\partial}{\partial x}\left(y^{n-2}\frac{\partial\Phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(y^{n-2}\frac{\partial\Phi}{\partial y}\right)=0.$$

The author considers the case $n-2=p$ for all positive values of the parameter p. The corresponding function Φ is called an axially symmetric potential in a space of $n=p+2$ dimensions. The corresponding potential theory is called generalized axially symmetric potential theory. From the equation for Φ the stream function $\Psi(x, y)$ can be obtained by replacing p by $-p$. Both Φ and Ψ satisfy the equations

$$(**)\qquad z_{xx}+z_{yy}+ky^{-1}z_y=0$$

for $k=p$ and $k=-p$ respectively. Writing $\Phi\{p\}$ and $\Psi\{p\}$ in order to emphasize the dependence upon the parameter p the important correspondence principle

$$\Psi\{p\}=Cy^{p+1}\Phi\{p+2\}\quad (C \text{ a constant})$$

can be proved. Thus the determination of the stream function $\Psi\{p\}$ in the axially-symmetric hydrodynamics of incompressible perfect fluids can be reduced to the determination of the electrostatic potential $\Phi\{p+2\}$ in a space of two more dimensions. A trivial example of this procedure is the determination of the flow about a sphere of unit radius which is usually obtained by putting a dipole at the origin. The problem is equivalent to finding the electrostatic potential of a five-dimensional sphere. Similarly, the flow problems about a spindle and a lens can be solved. Another interesting application of the method of generalized electrostatics is the application to the problem of the torsion of shafts. According to the correspondence principle this (five-dimensional) flow problem can be interpreted as an electrostatics problem in a space of seven dimensions. For positive values of y the differential equation (**) is elliptic with regular analytic coefficients. The x-axis is a singular line for these coefficients. There are some solutions which remain regular on the x-axis, for example, $z=x$. For such solutions some uniqueness and existence theorems have been proved by M. Hyman [cf. Bull. Amer. Math. Soc. 54, 1065 (1948) (abstract); 55, 284–285 (1949) (abstract)]. For positive k a uniqueness theorem had already been given by the author [Trans. Amer. Math. Soc. vol. 63, 342–354 (1948); these Rev. 9, 584].

One of the most interesting applications of the generalized axially-symmetric potential theory is a new approach to the theory of Tricomi's equation $\eta z_{\xi\xi}+z_{\eta\eta}=0$, $z=z(\xi, \eta)$, which is elliptic in the upper half-plane $\eta>0$ and hyperbolic in the lower half-plane $\eta<0$. If $\eta>0$, Tricomi's equation can be reduced by the transformation $x=\xi$, $y=2\eta^{\frac{3}{2}}$ $(\eta>0)$ to the equation

$$z_{xx}+z_{yy}+(3y)^{-1}z_y=0,$$

which is an axially-symmetric potential in $2\frac{1}{3}$ dimensions. Therefore, the results of the generalized axially-symmetric potential theory can be applied. *M. Pinl* (Dacca).

Rubinowicz, A. Fields defined by elementary laws. Acta Phys. Polonica 11, 155–178 (1952).

If a given scalar field u is isotropic and homogeneous, it is possible to define it by an elementary law in the form of a spherically symmetric field due to a simple source, $u=f(r)$, say. Its partial fields are then given by $f(r)$ and the partial derivatives of $f(r)$ with respect to the Cartesian coordinates x, y, z. The field of a partial derivative of order k may be

interpreted as the field of a multiple source of order 2^k, an idea which goes back to Maxwell and Sylvester.

By superimposing such fields, it is possible to obtain multiple fields $Y_{l,m}(\theta,\phi)R_l^k(r)$ of order 2^l (where k is an odd integer and $Y_{l,m}(\theta,\phi)$ is a spherical surface harmonic) which have in general a singularity at the origin. To obtain fields of the form $Y_{l,m}(\theta,\phi)S_l^k(r)$ which are finite at the origin, the field of a simple source is expanded by a bilinear formula involving both functions R and S, just as the generating function for the Legendre polynomials

$$\frac{1}{(r_1^2-2r_1r_2\mu+r_2^2)^{1/2}}=\sum_0^\infty P_n(\mu)\frac{r_2^n}{r_1^{n+1}}$$

involves the two harmonic functions $r^nP_n(\mu)$ regular at 0 and $r^{-n-1}P_n(\mu)$ with a singularity at 0.

The formulae expressing the multiple fields $Y_{l,m}(\theta,\phi)R_l^k(r)$ in terms of the fields of multiple sources can be used to write down all the multiple solutions of an equation $\sum_{s=0}^{n}a_s\Delta^s u=0$ (especially the potential equation $\Delta u=0$ and the wave equation $\Delta u+k^2u=0$), and also in certain cases the solutions regular at the origin. *E. T. Copson* (St. Andrews).

Fichera, Gaetano. Sul problema della derivata obliqua e sul problema misto per l'equazione di Laplace. Boll. Un. Mat. Ital. (3) **7**, 367–377 (1952).

Nouvelle interprétation de la condition de possibilité de Giraud [voir Bouligand, Giraud et Delens, Le problème de la dérivée oblique en théorie du potentiel, Hermann, Paris, 1935] dans le cas d'un domaine plan A limité par une courbe simple régulière Γ: on introduit $f(z)$ holomorphe dans A, avec, sur Γ, $\arg f(z)=\omega(z)$, angle de la normale en z à Γ avec la direction "oblique" $\lambda(z)$; pour que le problème en question soit possible, il faut et il suffit que la donnée-frontière $\partial u/\partial\lambda$ soit orthogonale à $|f(z)|$; cela permet d'étudier le cas "non régulier" (ω non strictement compris entre $-\pi/2$ et $\pi/2$) et en particulier le problème mixte. Le rapporteur signale une étude non sans rapports avec celle-ci par A. Liénard [J. Ecole Polytech. (3) **144**, 35–158 (1938)]. *J. Deny* (Strasbourg).

Graffi, Dario. Sul problema della derivata obliqua. Boll. Un. Mat. Ital. (3) **7**, 378–382 (1952).

L'étude des courants stationnaires dans une lame métallique soumise à un champ magnétique conduit à un problème de dérivée oblique; cela permet une interprétation physique de la condition de Giraud et Fichera et de la fonction $f(z)$ [voir l'analyse précédente]. *J. Deny* (Strasbourg).

Bononcini, Vittorio E. Sul problema di Dirichlet in domini rettangolari. Atti Sem. Mat. Fis. Univ. Modena **5**, 154–164 (1951).

Précision, et extension aux dérivées secondes mixtes, de résultats de L. Cesari [Rend. Circ. Mat. Palermo **60**, 185–212 (1936)] concernant la régularité des dérivées secondes pures de la solution du problème de Dirichlet relatif à un domaine rectangulaire A de R^n et à l'équation $\Delta u-\lambda u=f$, avec donnée-frontière continue; par exemple, si $f\in\operatorname{Lip}\alpha$, $0<\alpha\leq 1$, les dérivées secondes de $u\in\operatorname{Lip}\alpha'$, $0<\alpha'<\alpha$, sur tout compact de A. Comme dans l'article cité, on utilise les développements en série multiple de Fourier. *J. Deny.*

Karush, William, and Young, Gale. Temperature rise in a heat-producing solid behind a surface defect. J. Appl. Phys. **23**, 1191–1193 (1952).

Two boundary-value problems are solved for functions T harmonic in a semi-infinite region $z>0$ such that $T\to 0$ as $z\to\infty$. In the first problem the normal derivative $T_z(x,0)$ of the harmonic function $T(x,z)$ has a prescribed constant value over the infinite strip $x^2<a^2$ of the boundary $z=0$ and $T(x,0)=0$ outside that strip. This problem is solved by conformal mapping. In the second problem the normal derivative $T_z(r,0)$ of $T(r,z)$ has a prescribed constant value over a circular region $r<a$ of the boundary and $T(r,0)=0$ outside that region. This problem is solved by an analogy with the solution of the first. An application of the second problem to steady-state temperatures is discussed. *R. V. Churchill* (Ann Arbor, Mich.).

Tranter, C. J. Temperature rise in a heat-producing solid behind a surface defect. J. Appl. Phys. **24**, 369 (1953).

The solution of the first boundary value problem considered by Karush and Young in the paper reviewed above is found with the aid of a pair of dual integral equations. *R. V. Churchill* (Ann Arbor, Mich.).

Miyatake, Osamu, and Watanabe, Katsuo. Coulomb energy of a uniformly charged liquid-drop. J. Inst. Polytech. Osaka City Univ. Ser. B. Physics **2**, 15–22 (1951).

The electrostatic potential energy is calculated approximately for a uniformly charged volume whose boundary is a surface of revolution differing slightly from a sphere. *C. Strachan* (Aberdeen).

Differential Equations

Fort, Tomlinson. Reducibility of linear differential and difference equations. J. London Math. Soc. **28**, 156–163 (1953).

Let R be the real field. If f, g belong to the polynomial ring $R[x]$, if $0<\deg g<\deg f$ and if every root of g is a root of f, then f is reducible in $R[x]$. The present paper extends this definition to differential equations essentially as follows. If F, G are linear, homogeneous differential expressions in an unknown y, if the coefficients are real functions continuous on an interval and the initials are unity, if $0<\text{order } G<\text{order } F$, and if every solution of $G=0$ is a solution of $F=0$, then F is reducible. If order $G=j$, the order of reducibility of F is defined to be j. The definition is applied directly to give ten theorems on reducibility: for example, a third order equation is reducible of order one if and only if it has a solution without zero and reducible of order two if and only if it has a pair of solutions whose Wronskian is without zero. Reducibility for linear homogeneous difference equations is similarly defined with the additional restriction that the last coefficient be positive. *J. M. Thomas* (Durham, N. C.).

Germay, R. H. Sur une modalité de l'intégration par approximations successives des systèmes d'équations récurro-différentielles. Bull. Soc. Roy. Sci. Liège **21**, 403–407 (1952).

The author considers a system of differential equations of the form

$$\frac{dy_n^{(j)}}{dx}=F_n^{(j)}(x,\overset{(1)}{y_n},\overset{(2)}{y_n},\overset{(3)}{y_n},\overset{(1)}{y_{n+1}},\overset{(2)}{y_{n+1}},\overset{(3)}{y_{n+1}})$$

$$(j=1,2,3;\ n=1,2,3,\cdots),$$

with an associated system of initial conditions of the form

$$y_n^{(1)}(x_0)=a, \quad y_n^{(2)}(x_0)=b, \quad y_n^{(3)}(x_0)=c.$$

Assuming that the functions $F_n^{(j)}$ satisfy certain simple conditions as to boundedness and continuity, he proves that the unique solution is given, in a neighborhood of the point x_0, by a particular method of successive approximations.

L. A. MacColl (New York, N. Y.).

Garnier, René. Sur les systèmes différentiels Σ, à points critiques fixes, associés au problème de Riemann pour les systèmes linéaires d'ordre $m \geqq 2$. C. R. Acad. Sci. Paris 236, 161–164 (1953).

In an earlier work by the author [Ann. Sci. Ecole Norm. Sup. (3) 43, 177–307 (1926)] he obtained the solution of Riemann's problem for differential systems S_m (of order $m=2$) as a consequence of a study of the solutions of a differential system. The point of view is now reversed. In the complex plane x are given points t_i $(i=1, \cdots, n+3)$. Let C be a simple closed curve through the t_i, the points t_i and t_{i+1}, as well as t_{n+3} and t_1, being consecutive on C; designate by R^+ the interior and by R^- the exterior of C. Riemann's problem (R) is to find square matrices Φ^+ and Ψ^- of order m, analytic in R^+ and R^-, respectively, as well as on C, except perhaps on the t_i, where their elements are of the form $(x-t_i)^{r_{ij}}H_{jk}$, with H_{jk} analytic at $x=t_j$, so that (1) $\Phi^+=C_i\Psi^-$ on the arc t_it_{i+1}; here the C_i are assigned constant matrices, $|C_i|\neq 0$. One looks for solutions such that $|\Phi^+|$, $|\Psi^-|$ are nowhere zero. The solutions of (R) (when $t_{n+3}=\infty$) satisfy a matrix system (S)

$$dy/dx=y\sum_1^{n+2}(x-t_i)^{-1}A_i$$

[G. D. Birkhoff, Proc. Amer. Acad. Arts Sci. 49, 519–568 (1913)], where (Σ) $(t_k-t_i)dA^k/dt_i=A^iA^k-A^kA^i$ $(k\neq i)$, $\sum_{k=1}^{n+2}dA^k/dt_i=0$. The author obtains a particular solution of (Σ), as well as its representation near $t_n=0$; on the basis of this, with the aid of the author's work mentioned above, it is shown that the general solution of (Σ) has an analogous representation. This method can be extended to systems (S_m) $(m>2)$ and corresponding systems (Σ). For example, if C_{n+1}, C_{n-1}^{-1} have the canonical form $(e^{2\pi i\alpha_j}\delta_{jk})$ $(0<\alpha_1<\cdots<\alpha_m<1)$, the following holds: (1) t_n tending to zero, the A^k are analytic at $t_n^{\alpha_j-\alpha_k}$ and at $t_n^{1-\alpha_j+\alpha_k}$ $(\alpha_j>\alpha_k;\ j, h=1, \cdots, m)$; the A^k $(k\neq n, n+1)$ tend generally to finite limits and t_nA^n, t_nA^{n+1} tend to zero; (2) these properties belong to the general solution of (Σ). The part (1) can be treated effectively by successive approximations.

W. J. Trjitzinsky (Urbana, Ill.).

Markus, L. Invariant measures defined by differential equations. Proc. Amer. Math. Soc. 4, 89–91 (1953).

A brief derivation of results proved in more detail by the author elsewhere [cf. J. Math. Pures Appl. (9) 31, 341–353 (1952); these Rev. 14, 471]. *G. A. Hedlund.*

Duff, G. F. D. Limit-cycles and rotated vector fields. Ann. of Math. (2) 57, 15–31 (1953).

A "complete family of rotated vector fields" is a plane vector field $(P, Q)=F$, depending on a parameter α, satisfying suitable regularity conditions and such that: a) the critical points of F are isolated and independent of α; b) the argument θ of F is such that $\partial\theta/\partial\alpha>0$; and c) $F(\alpha+\pi)=-F(\alpha)$. The author gives a rather detailed description of the qualitative and quantitative changes undergone by the limit-cycles of F as α varies and derives new criteria for the existence and non-existence of limit-cycles of a (fixed) vector field. *J. L. Massera* (Montevideo).

Antosiewicz, H. A. Forced periodic solutions of systems of differential equations. Ann. of Math. (2) 57, 314–317 (1953).

The author considers the equation

$$\frac{dx}{dt}=A(t)x+p(x, t)+q(x, t, k), \tag{1}$$

where x is a vector in n-dimensional space E_n, under the following assumptions: (i) $A(t)$ is a continuous periodic matrix of period 1; (ii) $p(x, t)$ is continuous in (x, t) and periodic in t of period 1 for x in some fixed $\bar{S}$: $\|x\|\leqq R$; (iii) $p(0, t)\equiv 0$ and given any $\epsilon>0$ there exists a $\rho>0$ such that $\|p(x_1, t)-p(x_2, t)\|<\epsilon\|x_1-x_2\|$ for any x_1, x_2 in S_ρ: $\|x\|<\rho$ and any $t\geqq 0$; (iv) $q(x, t, k)$ is continuous in (x, t, k) and periodic in t of period 1 for x in $\bar{S}$, every t and every real k; (v) $q(x, t, 0)\equiv 0$ and $q(x, t, k)$ satisfies a Lipschitz condition with respect to x in $\bar{S}$. The author obtains the result that if all characteristic exponents of $dx/dt=A(t)x$ are less than unity in absolute value, then for sufficiently small values of k there exists at least one asymptotically stable periodic solution of (1) with period 1. This generalizes earlier results of other authors [see, e.g., Farnell, Langenhop, and Levinson, J. Math. Physics 29, 300–302 (1951); these Rev. 12, 706]. *C. E. Langenhop* (Ames, Iowa).

Gorbunov, A. D. On a method for obtaining estimates of the solution of a system of ordinary linear homogeneous differential equations. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 1950, no. 10, 19–26 (1950). (Russian)

Consider the system of linear homogeneous differential equations $dx/dt=E(t)x$ in which t is time, x is a real n-vector, and $E(t)$ is a square matrix of nth order, the elements of which are real, single-valued, and continuous functions of t in the interval $0<t<\infty$. By the method of successive approximations one may obtain a bound for the components of the solution in any interval $0<t<\tau$. But this bound goes to infinity as τ goes to infinity and therefore is not suitable for problems which are stable in the sense of Lyapunov. The object of this paper is the derivation of a formula which provides a bound more closely approximating the actual bounds for the case where x oscillates without increasing indefinitely. The author succeeds in finding a formula for such a bound provided it is possible to select a positive definite quadratic form in the n components of x which satisfies certain conditions. *W. E. Milne.*

Dyhman, E. I. On the reduction principle. Izvestiya Akad. Nauk Kazah. SSR 1950, no. 97, Ser. Mat. Meh 4, 73–84 (1950). (Russian)

Let x be an n-vector, y an m-vector and suppose that in the region Ω: $t>0$, $\|x\|<R$, $\|y\|<R$ they satisfy a system

$$\begin{aligned}\dot{y}&=\Phi(t; y)+A(t; x; y)\\ \dot{x}&=Px+G(t; x; y).\end{aligned} \tag{1}$$

Let $X=\|x_{sk}(t, t_0)\|$ satisfy $\dot{X}=PX$ and be such that $X(t_0, t_0)=I$. Suppose also that $|x_{sk}|<Be^{-\alpha(t-t_0)}$, where α, $B>0$ and do not depend on $t_0>0$. Suppose also that in Ω the components A_k of A satisfy: $|A_k|<A_1\eta^{N+1}$, where A is a constant and $\eta=\sup\{|y_k|\}$. Suppose that the solution $y=0$ of

$$\dot{y}=\Phi(t; y)+B(t; y) \tag{2}$$

is stable for all B in Ω such that $|B_k| < A\eta^{N+1}$ where A is a positive constant. Suppose finally that in Ω

$$|G_s| < M\{\|x\| + \|y\|\}^\beta, \quad M > 0,\ \beta > 1,$$

where M and β are constants. Theorem. The solution $x=0$, $y=0$ of (1) is stable. It is asymptotically stable [it is unstable] if $y=0$ is an asymptotically stable [an unstable] solution of (2).

The author points out that a stability criterion due to Malkin [Akad. Nauk SSSR. Prikl. Mat. Meh. **6**, 411–448 (1942), pp. 424–427; these Rev. **4**, 225] is erroneous. Two theorems correcting this criterion are given, but their statement is far too complicated to be reproduced here.

S. Lefschetz (Princeton, N. J.).

Dyhman, E. I. Some stability theorems. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 85–97 (1950). (Russian)

This paper contains further highly complicated stability theorems of which it will suffice to state the first. Given the system

$$\dot{y} = F(t; y), \tag{1}$$

$$\dot{x} = Px + Qy + \varphi(t; y) + \Phi(t; x; y), \tag{2}$$

suppose that the solution $y=0$ of (1) is unstable and that the matrix X behaves as in the preceding review. Let also in Ω: $\sum y_k^2 < R^2$, $t > 0$:

$$|F_k|,\ |\varphi_k| < L\sum |y_k|, \quad \Phi_s < D(\sum |x_k|)^\beta,$$

where L, D, β are constants and $\beta > 1$. Then $x=0$, $y=0$ is a stable solution of (1). [References: Persidskiĭ, Izvestiya Fiz.-Mat. Obščestva Naučno-Issled. Inst. Mat. Meh. Kazan. Univ. (3) **11**, 29–45 (1938); Četaev, Ustoĭčivost' dviženiya, Gostehizdat., Moscow-Leningrad, 1946; Goršin, Izvestiya Akad. Nauk Kazah. SSR **1948**, no. 56, Ser. Mat. Meh. **2**, 46–73; these Rev. **14**, 48]. *S. Lefschetz.*

Tulegenov, B. On stability of solutions of a system of differential equations of the second order. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 57–72 (1950). (Russian)

Let there be given a system

$$\begin{aligned} \dot{x}_1 &= p_{11}(t)x_1 + p_{12}(t)x_2 \\ \dot{x}_2 &= p_{21}(t)x_1 + p_{22}(t)x_2 \end{aligned} \tag{1}$$

where the coefficients $p_{ij}(t)$ are bounded and of weak variation. A number of properties of the characteristic numbers of Lyapunov are established with appropriate conclusions as to stability at the origin. Application is made to an equation $\ddot{x} + p(t)x = 0$. It is reduced to the equivalent system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -px_1, \tag{2}$$

and as a typical result it is shown that if

$$p(t) \leq a < 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \sqrt{-p}\ \ dt = b \neq 0,$$

then the system (2) is stable at the origin.

Extension is also made to a system

$$\begin{aligned} \dot{x}_1 &= p_{11}x_1 + p_{12}x_2 + X_1(t, x_1, x_2), \\ \dot{x}_2 &= p_{21}x_1 + p_{22}x_2 + X_2(t, x_1, x_2), \end{aligned} \tag{3}$$

where X_1, X_2 are small terms of higher order than $|x_1| + |x_2|$ and the p_{ij} are as in (1). The following result is proved. Let $D = 4p_{12}p_{21} + (p_{11} - p_{22})^2 \geqq a > 0$ and let the following limits exist

$$\lim_{t\to+\infty} \frac{1}{2t}\int_0^t (p_{11} + p_{22} \pm \sqrt{D})d\tau = \begin{cases}\alpha \\ \beta\end{cases}.$$

Then (3) is stable [unstable] whatever the X_i if both α, $\beta < 0$ [if one of α or $\beta > 0$]. *S. Lefschetz.*

Krasovskiĭ, N. N. On a problem of stability of motion in the large. Doklady Akad. Nauk SSSR (N.S.) **88**, 401–404 (1953). (Russian)

The author considers the system $dx/dt = f_1(x) + ay$, $dy/dt = bx + f_2(y)$ and derives simple sufficient conditions which ensure the asymptotic stability of the null solution $x = y = 0$ under arbitrary initial conditions. A typical condition is

$$\frac{f_1(x)}{x} + \frac{f_2(y)}{y} < 0, \quad \frac{f_1(x)}{x}\frac{f_2(y)}{y} - ab > 0 \quad \text{for} \quad x \neq 0,\ y \neq 0.$$

R. Bellman (Santa Monica, Calif.).

Dubošin, G. N. Some remarks on theorems of the second method of A. M. Lyapunov. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1950**, no. 10, 27–31 (1950). (Russian)

New proof of Lyapunov's first theorem on stability (sharpened to asymptotic stability). [See Lyapunov, Problème général de la stabilité du mouvement, Princeton, 1947; these Rev. **9**, 34]. *S. Lefschetz* (Princeton, N. J.).

Goršin, S. On Lyapunov's second method. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 42–50 (1950). (Russian)

Goršin, S. Some criteria of stability with constant disturbances. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 51–56 (1950). (Russian)

These two papers take up the application of Lyapunov's second method [see Problème général de la stabilité du mouvement, Princeton, 1947, p. 255; these Rev. **9**, 34] or the general idea of stability to an equation

$$\dot{x} = w(t; x) + f(t; x) \tag{1}$$

where x, w, f are vectors with a countable number of components, $w(t; 0) = 0$, w and f are continuous in a set $|x_n| < A$, $n = 1, 2, \cdots$, $t > \tau$ and in that set each $|f_n|$ can be made $< \rho$, any given positive number. In general, the author shows that the stability situation as obtained from Lyapunov's theorem or in general for (1) without f is unchanged by the presence of f. *S. Lefschetz* (Princeton, N. J.).

Eršov, B. A. On stability in the large of a certain system of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 61–72 (1953). (Russian)

The action of a certain automatic regulator is governed by a system

$$\begin{aligned} \dot{x} &= -Nax - by + \varphi(x, y), \\ \dot{y} &= f(cx - dy) = cx - dy + \psi(x, y), \end{aligned}$$

where a, b, c, d are positive constants and φ, ψ are the nonlinearities, $N = 1,\ 0,\ -1$ according as the system has positive self-correction, none or negative self-correction. It is very unclearly indicated that there is no limit-cycle; since the critical point at the origin appears to be the only one present and is stable, both x and $y \to 0$ as $t \to +\infty$. [Reference: Erugin, same journal **14**, 459–512 (1950); these Rev. **12**, 412.] *S. Lefschetz* (Princeton, N. J.).

Troickiĭ, V. A. On canonical transformations of the equations of the theory of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 49–60 (1953). (Russian)

Linear rearrangement without serious mathematical interest of a system

$$\dot{x}_k = \sum_{h=1}^{n} b_{kh} x_h + \sum_{\beta=1}^{m} c_{k\beta} f_\beta(x_1, \cdots, x_n).$$

S. Lefschetz (Princeton, N. J.).

Nemyckiĭ, V. V. Problems of the qualitative theory of differential equations. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **1952**, no. 8, 19–39 (1952). (Russian)

A most active seminar on the qualitative theory of differential equations has been conducted in Moscow since 1935 under the double guidance of Nemyckiĭ and Stepanov. It has already resulted in a noteworthy book on the subject published jointly by them and which has gone into a second much amplified edition [Qualitative theory of differential equations, Gos. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949; for a review of the 1st ed. (1947) see these Rev. **10**, 612]. The present article presents a detailed summary of the problems discussed and the results achieved in the last few years. The main topics dealt with are: Linear systems with constant coefficients (general form of the trajectories). Linear systems with variable coefficients: characteristic numbers of Lyapunov; stability; spectral theory. Study of $\ddot{x}+p(t)x=0$: boundedness of the solutions, locations of its zeros and of the zeros of $\ddot{x}+p\dot{x}+qx=0$. Method of variable frequency. Non-linear system in n variables. Asymptotic behavior of the solutions. Characteristic numbers of Lyapunov of the solutions. Asymptotically equivalent systems. Qualitative treatment of non-linear systems in the large. Investigation of limit cycles and other periodic solutions (systems of second order). A very ample bibliography concludes the summary. *S. Lefschetz* (Princeton, N. J.).

Persidskiĭ, K. On the stability of solutions of differential equations. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 3–18 (1950). (Russian)

Definition of derivative, integral, differential equation for $x(t)$ where x ranges over a Banach space. Proof of the existence theorem by successive approximations. Stability à la Lyapunov is discussed. Among the spaces included is the bounded subset of a general cartesian product of lines, metrized by $\sup|x_\alpha|$ (x_α any coordinate). *S. Lefschetz.*

Harasahal, V. On fundamental solutions of denumerable systems of differential equations. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 98–108 (1950). (Russian)

In this and the following four reviews the general topic and relevant references are the same. The references are as follows: Lyapunov, Problème général de la stabilité du mouvement, Princeton, 1947 [these Rev. **9**, 34]; Persidskiĭ, Izvestiya Akad. Kazah. SSR **1948**, no. 56, Ser. Mat. Meh. **2**, 3–35; Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 23–44, 635–650 (1950); Doklady Akad. Nauk SSSR (N.S.) **14**, 541–543 (1937); **63**, 229–232 (1948) [these Rev. **14**, 47; **11**, 520; **12**, 500; **10**, 299; see also the preceding review]; Harasahal, Izvestiya Akad. Nauk Kazah. SSR **1949**, no. 60, Ser. Mat. Meh. **3**, 77–84 [these Rev. **14**, 48]; Malkin, C. R. (Doklady) Acad. Sci. URSS (N.S.) **18**, 159–162, 162–164 (1938).

In all these papers the basic space is the bounded subset B of a countable normed space. If $x_1, x_2, \cdots$ are coordinates for an element x of B, then the norm is $\|x\| = \sup\{|x_n|\}$. One defines also $[x] = \sum|x_n|$. If T is the time interval $0 \leqq t < +\infty$, there is also considered in the product space $T\times B$the set E of functions $x(t)$ such that for any segment I_τ: $0\leqq t\leqq\tau$ there is an $N(\tau)>0$ such that $\|x(t)\|\leqq N(\tau)$ for any $t\in I_\tau$. [The vector language is the reviewer's throughout.]

Consider now a system

$$(1) \qquad \dot{x} = P(t)x$$

where the elements $p_k{}^s(t)$ of the matrix P are continuous for $t\geqq 0$ and if $p^s=(p_1{}^s, p_2{}^s, \cdots)$, then $[p^s]<A(t)$, $s=1, 2, \cdots$, where A is continuous for $t\geqq 0$. Existence and unicity of solutions of (1) in E through points of B are well known to hold [Persidskiĭ]. Theorem 1. Let $x^k(t, t_0)\in E$, $s=1, 2, \cdots$, be solutions of (1) such that $x_k{}^s(t_0, t_0)=a_k{}^s$ where $[a^s]=a_s$, $\|a\|<x<\infty$. Let $c\in B$ be constant with $\|c\|=\gamma$. Then the series $\sum_k c_k x_k{}^s$ are absolutely convergent and

$$\sum_k |c_k x_k{}^s| \leqq \alpha\gamma \exp \int_{t_0}^{t} A(t')dt'$$

for all s. The series of the derivatives are likewise absolutely convergent and $\sum_k|c_k\dot{x}_k{}^s|<\alpha\gamma A(t)\exp\int_{t_0}^t A(t)dt$. The function $x(t, t_0)$ with coordinates $x^s(t, t_0)=\sum_k c_k x_k{}^s(t, t_0)$ is a solution of (1) passing at time t_0 through $(a_1, a_2, \cdots)$, $a_s=\sum c_k a_k{}^s$. The series $x^s(t)$ are all absolutely convergent and termwise differentiable.

A fundamental system of solutions is then defined. Some sufficient conditions in order that $\|x_k{}^s\|$ of the theorem be a fundamental system are given in terms of $\|x_k{}^s(t_0, t_0)\|$.

S. Lefschetz (Princeton, N. J.).

Rešetov, M. On the boundedness of solutions and characteristic numbers of a denumerable system of linear differential equations of triangular form. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 109–114 (1950). (Russian)

The system under consideration is $\dot{x}=P(t)x$, P triangular. It is solved by successive approximations and one obtains again the bound of Persidskiĭ

$$\|x(t_0)\| \exp\left(-\int_{t_0}^{t} A(t')dt'\right) \leqq \|x(t)\|$$

$$\leqq \|x(t_0)\| \exp \int_{t_0}^{t} A(t')dt', \quad t\geqq t_0.$$

In particular, if $A(t)\leqq a$, the characteristic number α of the solution of (1) other than $x=0$ satisfies $|\alpha|\leqq a$.

S. Lefschetz (Princeton, N. J.).

Persidskiĭ, K. P. Some critical cases of denumerable systems. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 3–24 (1951). (Russian)

The system under consideration is

$$(1) \quad \dot{y}=F(t; x; y); \qquad (2) \quad \dot{x}=H(t; x; y)$$

$$F(t; 0; 0)=H(t; 0; 0)=0$$

$$\begin{matrix}\|F(t; x'; y')-F(t; x''; y'')\| \\ \|H(t; x'; y')-H(t; x''; y'')\|\end{matrix} < B(t)\|x'-x''; y'-y''\|,$$

where $B(t)$ is continuous. Typical theorem: If $y=0$ is stable for (1) when x is small and $x=0$ for (2) when y is small, then $x=0$, $y=0$ is stable for (1) and (2) together. A number of further complicated stability or instability theorems are displayed in the paper. *S. Lefschetz* (Princeton, N. J.).

Ibrašev, H. I. Some cases of stability of solutions of a denumerable system of differential equations. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 119–135 (1951). (Russian)

Extension of the Lyapunov "second method" stability theorems to a vector system in the space B:

$$\dot{x}=\omega(t;x),$$
$$\|\omega(t;x')-\omega(t;x'')\|\leqq A(t)\|x'-x''\|$$

where $A(t)$ is continuous for $t\geqq 0$. *S. Lefschetz.*

Harasahal, V. Stability in the first approximation of solutions of denumerable systems of differential equations. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 136–141 (1951). (Russian)

Consider a system

$$(1)\qquad \dot{x}=Px+L(t;x),\quad \|L\|\leqq B\|x\|^\beta,\quad B>0,\quad \beta>1.$$

Let σ be the spectrum of (1), i.e., the collection of its characteristic numbers (i.e., of the characteristic numbers of the norms of the solutions) in the sense of Lyapunov. Persidskiĭ has shown that σ could be finite, countable or uncountable. Let $\{x^k\}$, $x^k=(x_{1k}, x_{2k}, \cdots)$, be a fundamental system of solutions of the first approximation $\dot{x}=Px$ such that the matrix $\|x_{sk}(t_0, t_0)\|$ is the identity. Let us suppose that the spectrum of the solutions x^k is above a certain $\alpha>0$. Then there is a $D(t_0, \alpha)>0$ such that for any solution $x(t)$ of the first approximation: $\|x(t)\|\leqq x(t_0)D(t_0, \alpha)e^{-\alpha(t-t_0)}$ for all $t\geqq t_0$. Theorem. If for $t_0<0$: $D(\alpha, t_0)e^{-\gamma t_0}<M$, $0<\gamma<\alpha(\beta-1)$, then the solution $x=0$ of (1) is asymptotically stable whenever L is restricted as stated. *S. Lefschetz.*

Hartman, Philip. On the derivatives of solutions of linear, second order, ordinary differential equations. Amer. J. Math. **75**, 173–177 (1953).

Se $Q(t)$ è una funzione positiva e non decrescente e se $q(t)\leqq Q(t)$, per ogni integrale $y(t)$ dell'equazione $y''+qy=0$ si ha:

$$|y'(t)|\leqq 2Q^{\frac{1}{2}}(t+2/Q^{\frac{1}{2}}(t))m(t),$$

se $t\geqq 2/Q^{\frac{1}{2}}(0)$ e $m(t)=\max|y(u)|$ per

$$|t-u|\leqq 2/Q^{\frac{1}{2}}(t+2/Q^{\frac{1}{2}}(t)).$$

Se ne deducono vari corollari relativi allo spettro dell'equazione differenziale considerata. *C. Miranda.*

de Castro Brzezicki, A. Studies on nonlinear mechanics. I. On the general differential equation of relaxation oscillations. Revista Mat. Hisp.-Amer. (4) **12**, 266–280 (1952). (Spanish)

It is shown that the equation $\ddot{x}+f(x, \dot{x})\dot{x}+G'(x)=0$ possesses a periodic solution if the functions f and G satisfy the following set of conditions: (1) f and G' are continuous and satisfy Lipschitz conditions in every bounded domain; (2) if $x\neq 0$, then $xG'(x)>0$; (3) $G(\pm\infty)=\infty$; (4) $f(0, 0)<0$; (5) if $|x|\geqq x_0$ and $|v|\geqq a$, then $f(x, v)>0$; (6) if $|x|>x_0$ and $|v|\leqq a$, then $f(x, v)\geqq -M$ and $f(x, v)+f(x, -v)\geqq 0$; (7) there exists a number $x_1>x_0$ such that if $v(x)$ is a decreasing function satisfying the condition $v>a$, then

$$\int_{x_0}^{x_1} f(x, v)dx\geqq 4Mx_0+\alpha.$$

(The symbols x_0, a, M, α denote positive constants.) Two somewhat more special sets of conditions, sufficient for the existence of a periodic solution, are also given. The paper concludes with a consideration of some conditions under which the periodic solution can be shown to be unique. *L. A. MacColl* (New York, N. Y.).

Staržinskiĭ, V. M. On the stability of a mechanical system with one degree of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 117–122 (1953). (Russian)

The author considers the problem of determining the set of a-values for which the solutions of $y''+ay'+p(x)y=0$, where $p(x)$ is periodic, approach zero as $x\to\infty$. As an application he determines the first two stability domains, in the (a, c)-plane, of the equation $y''+ay'+c(1+\lambda\cos x)y=0$, $0<\lambda<1$. The method employed is an extension of that of Liapounoff, who considered the case $a=0$. *R. Bellman.*

Colombo, Giuseppe. Sopra un fenomeno di isteresi oscillatoria. Rend. Sem. Mat. Univ. Padova **21**, 370–382 (1952).

The purpose of this paper is to give a mathematically rigorous treatment of the oscillation hysteresis in a triode generator with two degrees of freedom discussed previously by van der Pol [Philos. Mag. (6) **43**, 700–719 (1922)]. This involves the study of the periodic solutions of a system of differential equations of the form

$$\ddot{x}+x=\epsilon my+\epsilon\frac{d}{dt}(\alpha x-\gamma x^3),\quad \ddot{y}+y=\epsilon nx-\epsilon h\dot{x}-\epsilon\zeta y,$$

where ϵ is a small positive parameter, and m, n, α, γ, h, ζ are constants, the first four of which are positive. By means of methods developed by the author in a paper to be published it is shown that there exist two solutions periodic to within terms of order $O(\epsilon^2)$. Their domains of existence, in the space of the parameters of the problem, overlap partly. At least one of these solutions is stable, in first approximation with respect to ϵ. In certain subregions they are both stable. The values of the parameters at which the system jumps from one frequency to the other depends on the previous values of these parameters. *W. R. Wasow.*

★Wu, M. H. Lee. Subharmonic resonance of system having non-linear spring with variable coefficient. Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 147–153. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Košlyakov, N. S. Investigation of a class of differential equations with doubly periodic coefficients. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 537–562 (1952). (Russian)

The author considers the generalized Lamé equation

$$u''+\left[h-k^2n(n+1)\,\mathrm{sn}^2x-\frac{\lambda(\lambda-1)}{\mathrm{sn}^2x}-\frac{\mu(\mu-1)\,\mathrm{dn}^2x}{\mathrm{cn}^2x}-\frac{\nu(\nu-1)k^2\,\mathrm{cn}^2x}{\mathrm{dn}^2x}\right]u=0$$

and tries to find solutions of the form

$$u=\mathrm{sn}^px\cdot\mathrm{cn}^qx\cdot\mathrm{dn}^rx\cdot P(\mathrm{sn}^2x),$$

where p, q, r are rational and P is a polynomial. It is shown that if such solutions exist, they satisfy a homogeneous integral equation of the second kind whose kernel depends on a certain hypergeometric polynomial. This method is applied to obtain solutions in a great number of particular cases and finally to the investigation of the fundamental tone of the vibration of a string lying on the surface of a sphere. *J. L. Massera* (Montevideo).

Levi, Beppo. On the solution of nonhomogeneous linear differential equations. Math. Notae 12–13, 1–18 (1952). (Spanish)

This paper describes a method for solving an inhomogeneous linear differential equation which is essentially a variant of Lagrange's method of the variation of the constants. The author considers that the new method possesses advantages in the way of directness and freedom from artificiality. However, it appears to the reviewer that these advantages, if they exist, are only slight.

L. A. MacColl (New York, N. Y.).

Levinson, Norman. Certain explicit relationships between phase shift and scattering potential. Physical Rev. (2) 89, 755–757 (1953).

An intuitive account is given of a method of Gel'fand and Levitan [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951); these Rev. **13**, 558] for the determination of the potential in the one-dimensional Schrödinger equation from the asymptotic phase shift at infinity, the bound states, and the normalizing factors of the eigenfunctions. In this method the kernel of a certain Fredholm integral equation involving a parameter is calculated from the data, and the potential is found by solving this integral equation for many values of the parameter and then differentiating the solution.

The author shows that the method can also be used to obtain the first variation of the potential corresponding to a small change in the phase shift, bound states, and normalizing factors.

W. Wasow (Los Angeles, Calif.).

Miller, Kenneth S., and Schiffer, Menahem M. Monotonic properties of the Green's function. Proc. Amer. Math. Soc. 3, 948–956 (1952).

Gli A. considerano un'equazione differenziale lineare ordinaria di ordine $n=2r$, assegnando agli estremi di un intervallo i valori della funzione incognita e delle sue prime $r-1$ derivate e dimostrano che certe forme quadratiche, i cui coefficienti sono formati a partire dalla funzione di Green del problema sono definite, positive o negative secondo che r è dispari o pari. Gli A. si valgono di alcuni risultati, da essi precedentemente stabiliti, relativi alle variazioni infinitesime della funzione di Green [stessi Proc. **3**, 433–441 (1952); questi Rev. **14**, 50].

C. Miranda.

Conti, Roberto. Determinazione esplicita, in funzione dei dati, del nucleo della equazione integrale traducente un problema ai limiti. Estensione ai sistemi di equazioni differenziali di un procedimento di G. Cimmino. Boll. Un. Mat. Ital. (3) 7, 396–403 (1952).

In the differential expression $L_n[u]=\sum_{s=0}^{n}P_s(t)d^{n-s}u/dt^{n-s}$ let u be a k-dimensional column vector and $P_s(t)$ a square matrix. Starting from Green's formula linking $L_n[u]$ with its adjoint $M_n[v]=\sum_{s=0}^{n}(-1)^{n-s}(d^{n-s}/dt^{n-s})(P_s'v)$, ($P_s'$ is the transpose of P_s), a Volterra integral equation is derived for the solution of the vector differential equation $M_n[v]=g(t)$ satisfying certain boundary conditions. If $v_i(t)$ $(i=1, \cdots, k)$ are the components of $v(t)$, these boundary conditions are of the form $v_i(t_{ij})=a_{ij}$ $(j=1, \cdots, n)$, where $t_{i\alpha}\neq t_{i\beta}$ for $\alpha\neq\beta$, and the a_{ij} are given constants. As in Cimmino's paper mentioned in the title [Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **6**, 205–225 (1950); these Rev. **12**, 336] this integral equation differs from those using Green's function in that it involves only the original data of the problem and simple explicit rational functions. If $P_0(t)$ is the identity matrix, then the kernel of the integral equation is a polynomial in the t_{ij}.

W. R. Wasow.

Povzner, A. Ya. On the expansion of arbitrary functions in characteristic functions of the operator $-\Delta u+cu$. Mat. Sbornik N.S. 32(74), 109–156 (1953). (Russian)

Let L be a self-adjoint extension of the differential operator $T=-\Delta+c$ where Δ is Laplace's operator in three-space and $c=c(p)$ is a continuous function. The domain of definition of T consists of all sufficiently differentiable functions with compact supports. It is shown that the resolvent R_ζ and the associated projections $E_\lambda-E_0$ of L are integral operators with kernels of the Carleman type [Ark. Mat. Astr. Fys. **24B**, no. 11 (1934)] and that the kernel $\vartheta(p, q, \lambda)$ of $E_\lambda-E_0$ is continuous in p, q and differentiable with respect to a measure τ so that $d\vartheta(p, q, \lambda)=\psi(p, q, \lambda)d\tau(\lambda)$, where ψ is continuous for almost all λ. [This result has been announced earlier in Doklady Akad. Nauk SSSR (N.S.) **79**, 193–196 (1951); these Rev. **13**, 241.] If T is bounded from below, then already its closure is self-adjoint and the kernel of R_ζ is by means of a Laplace transform connected with the solution w of the wave equation $w_{tt}-\Delta w+cw=0$ with the initial conditions $w=0$, $w_t=\delta(p-p_0)$ $(t=0)$. The second part of the paper deals chiefly with the case when $c(p)=O(|p|^{-3-\epsilon})$ $(\epsilon>\frac{1}{2})$. Then τ is absolutely continuous outside the discrete spectrum which is situated on the positive axis. There is an application to the scattering problem: find a solution u of $Tu=\lambda^2u$ of the form $e^{i\lambda(\omega,p)}+v$ where $rv=O(1)$ and $r(v_r-i\lambda v)=o(1)$ as $r=|p|\to\infty$. If λ^2 is not in the discrete spectrum, the problem has a unique solution.

L. Gårding (Lund).

Gillis, Paul P. Sur la primitive d'une forme différentielle extérieure fermée. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 612–631 (1952).

In the Euclidean space R^n $(n>2)$ let D^n be a closed and bounded domain which is simply connected. Let c^p be used to denote any p-dimensional variety in D^n, and let fc^p denote the boundary of c^p. The coefficients of the symbolic form

$$w_p=\sum a^{\alpha_1\cdots\alpha_p}dx_{\alpha_1}\cdots dx_{\alpha_p}$$

$$(dx_jdx_k=-dx_kdx_j;\ a^{jk\alpha_3\cdots\alpha_p}=-a^{kj\alpha_3\cdots\alpha_p}, \text{etc.})$$

are assumed to be continuous functions of n variables $x_1, \cdots, x_n$. The symbolic form

$$w_p^*=\sum\pm a^{\alpha_1\cdots\alpha_p}dx_{\alpha_{p+1}}\cdots dx_{\alpha_n}$$

is called the adjoint of w_p. Here one selects the $+$ sign if $\alpha_1\cdots\alpha_n$ is an even permutation of $12\cdots n$, otherwise one uses the $-$ sign. If for every c^{p+1} in D^n the $\int_{fc^{p+1}}w_p=0$, then w_p is said to be closed in D^n. If w_p is closed in D^n, there exist forms w_{p-1}, called primitives of w_p, such that $\int_{fc^p}w_{p-1}=\int_{c^p}w_p$ for each c^p in D^n. In the present paper the author shows that if w_p is closed in D^n, it has a primitive w_{p-1} whose adjoint w^*_{p-1} is also closed in D^n.

For a closed symbolic Hermitian form

$$\sum(A^{jk}+iB^{jk})(dx_j+idy_j)(dx_k-idy_k)\quad(A^{jk}=A^{kj},\ B^{jk}=-B^{kj})$$

it is shown that there exists a function

$$V=V(x_1, \cdots, x_n; y_1, \cdots, y_n)$$

such that

$$A^{kj}=\frac{\partial^2V}{\partial x_k\partial x_j}+\frac{\partial^2V}{\partial y_k\partial y_j},\quad B^{jk}=\frac{\partial^2V}{\partial x_k\partial y_j}-\frac{\partial^2V}{\partial y_k\partial x_j}.$$

F. G. Dressel (Durham, N. C.).

Moses, H. E. A note on the application of Schwinger's variational principle to Dirac's equation of the electron. Quart. Appl. Math. 11, 111–118 (1953).

Dirac's equation for an electron in an electromagnetic field is written with the electromagnetic potentials as forcing terms; thus the author obtains an integral equation for the scattered wave produced by an incident plane wave. The author sets up variational expressions for the amplitudes by using the following well-known device: Let a, x, b, y be vectors in a Hilbert space and K, K^* be adjoint operators. If (1) $Kx=a$, (2) $K^*y=b$, then the expression $S=(y', Kx')(y', a)^{-1}(b, x')^{-1}$ is stationary for variations of x', y' about the solutions of (1) and (2) and the stationary value of S is $(b, x)^{-1}$. *B. Friedman* (New York, N. Y.).

Borowitz, Sidney, and Friedman, Bernard. Variational principles for three-body scattering problems. Physical Rev. (2) 89, 441–445 (1953).

Schwinger's variation principle for two-body problems is here generalized to three-body collisions. The case when the scattered particle is identical with one of the two particles in the scatterer is particularly treated, and several stationary expressions for the direct and the exchange scattered amplitudes are derived. The authors also point out and discuss some peculiarities in the ordinary expansion of a plane wave in the complete set of eigenfunctions of a Hamiltonian having a spherically symmetric potential. *P. O. Löwdin.*

Dingle, R. B. The solution of the Schrödinger equation for finite systems, with special reference to the motion of electrons in Coulomb electric fields and uniform magnetic fields. Proc. Cambridge Philos. Soc. 49, 103–114 (1953).

The solution of the Schrödinger equation for systems of finite extent, having eigenfunctions vanishing at a given boundary, etc., is investigated. A number of methods are developed in detail for treating the case of an electron moving in a Coulomb field within a sphere of given radius and for the case of an electron moving in a uniform magnetic field within a cylinder. Several special cases connected with various physical problems are discussed. *P. O. Löwdin.*

Moon, Parry, and Spencer, Domina Eberle. Recent investigations of the separation of Laplace's equation. Proc. Amer. Math. Soc. 4, 302–307 (1953).

The authors have previously defined [same Proc. 3, 635–642 (1952); these Rev. 14, 173] an equation to be separable, for Euclidean 3-space and curvilinear coordinates, if the assumption

$$\Phi=U^1(u^1)U^2(u^2)U^3(u^3)/R(u^1, u^2, u^3)$$

permits separation into three ordinary differential equations. This definition is compared with another given by Levinson, Bogert and Redheffer [Quart. Appl. Math. 7, 241–262 (1949); these Rev. 11, 251]. The LBR definition requires that the variables be split off one at a time. The latter is the more restrictive of the two definitions. For example, Laplace's equation in paraboloidal coordinates cannot be separated according to LBR. An error in theorem (VII) of the LBR paper is pointed out. *T. E. Hull.*

Courant, Richard, Isaacson, Eugene, and Rees, Mina. On the solution of nonlinear hyperbolic differential equations by finite differences. Comm. Pure. Appl. Math. 5, 243–255 (1952).

In this paper the authors discuss two schemes for obtaining the solution of a system of quasi-linear hyperbolic differential equations in two independent variables by finite differences. In the first a rather general curvilinear mesh system is used, while the second is based on a rectangular net. In both instances the replacement of derivatives by finite differences is made along characteristic directions. In general, convergence is established provided the conditions that the domain of dependence of any point in the mesh as given by the finite difference equations is not less than the domain of dependence determined by the differential equations. The effect of round-off is also taken into consideration in the proofs. *H. Polachek* (Carderock, Md.).

Schmidt, Adam. Existenz, Unität und Konstruktion der Lösung für das Anfangswertproblem bei gewissen Systemen quasilinearer partieller Differentialgleichungen. Math. Nachr. 7, 261–287 (1952).

The hyperbolic systems considered are of the form

$$\frac{dy}{dt}+A\frac{dy}{dx}=0,$$

where y is a column vector and A a square matrix depending on y and the independent variables x, t. The values of y are prescribed for $t=0$. The existence and uniqueness theorems proved here are essentially the same as those derived by Courant and Lax [Comm. Pure Appl. Math. 2, 255–273 (1949); these Rev. 11, 441] and by Beckert [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 97, no. 5 (1950); these Rev. 12, 415]. The method of proof is more closely related to that used in the recent paper by Courant, Isaacson, and M. Rees [see the paper reviewed above]. Approximations for the values of a solution are determined successively on a discrete set of lines $t=$const. The values at a point of one line are determined by drawing approximate characteristics from that point to the preceding line, and solving the resulting difference system. The precise regularity assumptions under which convergence of this procedure to a solution is proved are too involved to be quoted here, but they include existence of bounded second derivatives of the eigenvalues and eigenvectors of the matrix A. *F. John* (New York, N. Y.).

Fourès-Bruhat, Y. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. Acta Math. 88, 141–225 (1952).

The paper aims at establishing existence of solutions with given (non-analytic) initial data for a class of differential equations which is sufficiently general to include the gravitational equations of relativity theory. The author considers a system of n second-order differential equations of the form

$$(1)\qquad A^{\lambda\mu}\frac{\partial^2 W_s}{\partial x^\lambda \partial x^\mu}+f_s=0 \quad (s=1, \cdots, n)$$

for n functions W_s of four independent variables x^i. First the linear case is taken up, in which the $A^{\lambda\mu}$ are given functions of the x^i alone, and the f_s are linear in the first derivatives of the W_s with coefficients which depend only on the x^i. Following Sobolev [Mat. Sbornik N.S. 1(43), 39–70 (1936)] and Christianovich [ibid. N.S. 2(44), 871–899 (1937)] a "formula of Kirchhoff" is derived, in which the value of a solution at a point M is expressed in terms of integrals of the solution over the surface of the characteristic conoid with vertex M, and in terms of the initial data. Thus the Cauchy problem for the linear differential equations is seen to lead to integral equations. Next the quasi-linear case is considered, in which the $A^{\lambda\mu}$ depend on the x^i and W^s, and

the f_i depend on the x^i, the W^i and the first derivatives of the W^i. Here the W_i occurring in the A^{hp} are first replaced by functions $W_i^{(1)}$. Differentiating the resulting system (1) four times one obtains a Cauchy problem for a system of differential equations in the W_i and their derivatives up to the fourth order. This system can be solved by the reduction to integral equations applicable to the linear case. The resulting mapping of functions $W_i^{(1)}$ into functions W_i can be shown to have a fixed point. The fact that the solution of the integral equations satisfies the differential equations is established by approximation by equations with analytic coefficients. The resulting existence theorem can be applied to the gravitational equations $R_{\alpha\beta}=0$, which are of the form (1) when written in isothermic coordinates. It is proved that there exists a solution consistent with the conservation laws for given initial data, and that that solution is unique within coordinate transformations. The author indicates the procedure to be followed in the more general case of equations of the form (1), in which the A^{hp} contain in addition to the x^i and W_i also the first derivatives of the W_i.

F. John (New York, N. Y.).

Yosida, Kôsaku. On Cauchy's problem in the large for wave equations. Proc. Japan Acad. **28**, 396–403 (1952).

The author considers the problem of integrating, in the large, the wave equation (*) $\partial^2u(x,t)/\partial t^2=Au(x,t)$ with the initial conditions (**) $u(x,0)=g(x)$, $\partial u(x,0)/\partial t=h(x)$. Here x is a point in a connected domain R of an orientable m-dimensional Riemann space, and A is a second order elliptic differential operator on x which is defined with respect to the metric of R, and is formally self-adjoint in the old sense of the word and non-positive definite. The functions g and h are twice continuously differentiable and have a compact support. The first step consists in integrating the equation formally in the sense of the operational calculus in Hilbert space. For this purpose let L be the set of twice continuously differentiable functions in R with compact support and satisfying a boundary condition such that for $f, v \in L$ one has

$$\int f(Av)=\int (Af)v,\quad \int (Af)f\leqq 0.$$

Let L^* be the Hilbert space obtained by completion of L under the classical Hilbert space norm. Following Friedrichs [Math. Ann. **109**, 465–487 (1934)] the author constructs a non-positive definite self-adjoint extension $\bar{A}$ of A in L^*. Using the spectral resolution of $-A$ and its positive square root one obtains an operational solution $\bar{u}(x,t)$ of the problem. This is simply the proper Hilbert space interpretation or analogue of the classical "formal" solution by Fourier integrals.

The essential step now consists in proving the following theorem. Let D be the subset of L such that $f \in D$ if and only if f is infinitely differentiable and in the domain of A^q for each $q>0$. If then $g, h \in D$, the solution $\bar{u}(x,t)$ is equivalent (in the (x,t)-space with the obvious product measure) to a function $u(x,t)$ which is a genuine solution of our problem and such that $u(x,t) \in D$ for each fixed t. To prove this theorem the author uses a parametrix representation of $\bar{u}$. The trouble is that he is obliged first to construct a parametrix for the operator A. He does this for general elliptic operators (not necessarily formally self-adjoint ones) by adapting a construction previously used by M. Riesz [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713], and using the method of descent from odd to even numbers of dimension.

W. Feller (Princeton, N. J.).

Seth, B. R. Some solutions of the wave equation. Proc. Indian Acad. Sci., Sect. A. **32**, 421–423 (1950).

Let $\phi(x,y)$ satisfy the equation $\phi_{xx}+\phi_{yy}+p^2\phi=0$ in a region and the condition $\phi=k$ on the boundary of the region, where p and k are constants. For certain regions with rectilinear boundaries, including rectangles, right isosceles and equilateral triangles, the function ϕ is written in terms of finite or infinite trigonometric series.

R. V. Churchill.

Karol′, I. L. On a boundary problem for an equation of mixed elliptic-hyperbolic type. Doklady Akad. Nauk SSSR (N.S.) **88**, 197–200 (1953). (Russian)

The Tricomi problem is discussed for the equation

$$(*)\qquad u_{xx}+\operatorname{sign} y\,|y|^m u_{yy}=0,\quad 0<m<1.$$

The domain considered consists of the normal curve Γ: $y=(1-2\beta)^{2\beta-1}[x(1-x)]^{\frac{1}{2}-\beta}$, $0\leqq x\leqq 1$, where $\beta=m/2(m-2)$, and the two characteristics which issue from the points $(0,0)$, $(1,0)$, respectively, which intersect. Boundary values are prescribed on Γ and on one of the characteristics. By following the methods Tricomi employed for the case $m=-1$, the author sets up an integral equation for $u_y(x,0)$, and obtains an explicit solution. The uniqueness follows from the formulas obtained. Since the characteristics are tangent to the x-axis rather than normal as in the case $m<0$, the boundary values are required to satisfy an additional restriction at the boundary points $(0,0)$ and $(1,0)$. The solution obtained is a strict solution in the elliptic portion of the domain and a "generalized" solution in the hyperbolic part.

M. H. Protter (Berkeley, Calif.).

Karol′, I. L. On the theory of equations of mixed type. Doklady Akad. Nauk SSSR (N.S.) **88**, 397–400 (1953). (Russian)

This note continues the work started earlier [see the above review for notation] and discusses the equation

$$u_{xx}+yu_{yy}+\alpha u_y=0,$$

where α is a real constant. By a change of variable this equation can be reduced to (*) with the correspondences: $0<\alpha<\frac{1}{2}$ corresponds to $0<m<1$, $\alpha=\frac{1}{2}$ to $m=0$, and $\frac{1}{2}<\alpha<1$ to $m<0$. Then the solution of the Tricomi problem for $\frac{1}{2}<\alpha<1$ is reduced to a problem solved by Gellerstedt [Thesis, Uppsala, 1935]; for $\alpha=\frac{1}{2}$ it is reduced to a problem solved by Lavrent′ev and Bicadze [same Doklady (N.S.) **70**, 373–376 (1950); these Rev. **11**, 724]; and the case $0<\alpha<\frac{1}{2}$ is reduced to equation (*) above. For the case $\alpha<0$ the following problem is solvable: boundary values assigned along Γ and both characteristics, i.e., the Dirichlet problem. Again a restriction must be placed on the boundary values at $(0,0)$ and $(1,0)$. For $\alpha>1$ the solution in general becomes infinite on the x-axis.

M. H. Protter.

Reuter, G. E. H., and Ledermann, W. On the differential equations for the transition probabilities of Markov processes with enumerably many states. Proc. Cambridge Philos. Soc. **49**, 247–262 (1953).

Kolmogorov's equations

$$(1)\qquad \partial p_{ik}(s,t)/\partial t=\sum_j p_{ij}(s,t)a_{jk}(t)$$

$$(i,j=1,\cdots,n,\cdots;\ t\geqq s)$$

$$(2)\qquad -\partial p_{ik}(s,t)/\partial s=\sum_j a_{ij}(s)p_{jk}(s,t)$$

with the respective initial conditions $p_{ik}=\delta_{ik}$ at $t=s$ and at $s=t$ are integrated under the condition that the coefficients

a_{ij} are continuous. As the infinitesimal transition probabilities, these a_{ij} are assumed to satisfy $a_{ij}(t) \geqq 0$ $(i \neq j)$, $a_{ii}(t) \leqq 0$ and $\sum_j a_{ij}(t) = 0$. The authors intend to give a rather elementary derivation of Feller's results [Trans. Amer. Math. Soc. 48, 488–515 (1940); these Rev. 2, 101]. Thus they take up the nth section, that is, the differential system restricted to $i, j \leqq n$ and then they let n tend to ∞. In this way a solution $f_{ik}(s, t)$ is obtained satisfying

$$f_{ik}(s,t) \geqq 0, \quad \sum_j f_{ij}(s,t) \leqq 1, \quad f_{ij}(s,t) = \sum_k f_{ik}(s,u) f_{kj}(u,t) \qquad (s \leqq u \leqq t)$$

and such that $p_{ik}(s, t) \geqq f_{ik}(s, t)$ for any continuous solution $p_{ik}(s, t) \geqq 0$. The non-uniqueness [Doob, ibid. 58, 455–473 (1945); these Rev. 7, 210; and Feller, loc. cit.] of the solution of the Backward Equation (2), which corresponds to the case $\sum_k f_{ik}(s, t) < 1$, is also discussed. In particular, the problem is discussed in connection with birth-and-death processes: the case of constant matrix (a_{ij}) such that $a_{ij} = 0$ when $|i-j| > 1$. *K. Yosida* (Osaka).

***Hille, Einar. On the integration problem for Fokker-Planck's equation in the theory of stochastic processes.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 183–194. Johan Grundt Tanums Foriag, Oslo, 1952. 27.50 kr.

The author formulates the integration problem of Fokker-Planck's equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 b(x) U}{\partial x^2} - \frac{\partial a(x) U}{\partial x} \quad (t > 0, \ -\infty < x < \infty; \ b(x) > 0)$$

as the Problem A: What conditions should $a(x)$ and $b(x)$ satisfy in order that for every $g(x) \in C^2(-\infty, \infty)$ with compact carrier the above equation shall have a unique solution $U(x, t)$ with the following properties:

i) $$\lim_{t \downarrow 0} \int_{-\infty}^{\infty} |U(x,t) - g(x)| dx = 0;$$

ii) $U(x, t)$ is non-negative and $\int_{-\infty}^{\infty} U(x, t) dx = \int_{-\infty}^{\infty} g(x) dx$ if $g(x)$ is non-negative. [Cf. the reviewer, J. Math. Soc. Japan 1, 244–253 (1949); these Rev. 12, 190.] By virtue of the semi-group theory due to the author and the reviewer [Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; these Rev. 9, 594; Yosida, J. Math. Soc. Japan 1, 15–21 (1948); these Rev. 10, 462], this problem is replaced by the Problem B: What conditions should $a(x)$ and $b(x)$ satisfy in order that for every $g(x) \in L(-\infty, \infty)$ the equation $d^2 b(x) Y/dx^2 - da(x) Y/dx - \lambda Y = -g(x)$ shall, for $\lambda > \lambda_0 \geqq 0$, have a unique solution $Y(x, \lambda) \in L(-\infty, \infty)$ such that $Y(x, \lambda)$ is non-negative and $\int_{-\infty}^{\infty} Y(x, \lambda) dx = \int_{-\infty}^{\infty} g(x) dx$ when $g(x)$ is non-negative. It is proved that these problems are solvable if (I) $\sup_x (b''(x) - a'(x)) < \infty$ and

(II) $$\sup_x [(|a(x)| + |b'(x)|)/(|x|+1)] < \infty.$$

Moreover, under the condition (I), a necessary and sufficient condition is given for the solvability of the problems. In particular, the problems are solvable if either (i) $a(x) = 0$, $\sup_x b''(x) < \infty$, or

(ii) $$a(x) = b'(x), \quad \int_0^{\infty} t b(t)^{-1} dt = -\int_{-\infty}^{0} t b(t)^{-1} dt = \infty,$$

or (iii) $a(x) = b(x)$, (I) and $\int_0^{\infty} b(t)^{-1} dt = \int_{-\infty}^{0} b(t)^{-1} dt = \infty$.

K. Yosida (Osaka).

v. Szalay, L. Anwendung des Picone-Verfahrens auf die Wärmeleitungsdifferentialgleichung. Elektrotech. Z. 74, 141–143 (1953).

The author gives a method of solution of the nonhomogeneous heat equation for which the source function is dependent upon both time and position. The method depends upon a Green's formula and the fact that the solution can be developed in terms of a set of orthogonal eigenfunctions in the domain of definition of the problem. An example is given. *C. G. Maple* (Ames, Iowa).

Vacca, Maria Teresa. Conduzione del calore in una piastra anulare, sottile, limitata da due circonferenze concentriche. Atti Sem. Mat. Fis. Univ. Modena 5, 190–212 (1951).

The method of separation of variables and the Laplace transform are combined to solve a heat conduction problem for the annulus $A: r_1 < r < r_2$. Here $r^2 = x^2 + y^2$ and $r_1 > 0$. The problem is to find a function $u = u(r, \theta, t)$ satisfying in A and for $t > 0$ the differential equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} - k u_t = a(u - \alpha),$$

and in addition to meet the initial and boundary conditions

$$\lim_{t \to 0} u = f(r, \theta), \quad u(r_j, \theta, t) = f_j(\theta, t) \quad (j = 1, 2).$$

In the above relations k, a, α are positive constants, and f, f_j are continuous functions. The functions $f_j(\theta, t) = f_j(\theta + 2\pi, t)$ are assumed to remain bounded as $t \to +\infty$. The question of uniqueness of the solution is also treated.

F. G. Dressel (Durham, N. C.).

Rainville, Earl D. A heat conduction problem and the product of two error functions. J. Math. Physics 32, 43–47 (1953).

Let R be the region $x > 0$, $y > 0$, $t > 0$, and consider the following heat conduction problem. The function $u = u(x, y, t)$ is to satisfy in R the conditions: $u_t = u_{xx} + u_{yy}$, $\lim_{t \to 0} u = 1$, $\lim_{x \to 0} u = 0$, $\lim_{y \to 0} u = 0$. By solving this problem in two different ways the author arrives at the following representation of the product of two error functions

$$\operatorname{erf}(r \cos\theta) \operatorname{erf}(r \sin\theta) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(4n+2)\theta] r^{4n+2} {}_1F_1(2n+1; 4n+3; -r^2)}{(2n+1) 2^{4n+2} (3/2)_{2n}},$$

$r > 0$, $0 \leqq \theta \leqq \pi/2$. Here

$$\operatorname{erf}(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-\alpha^2} d\alpha, \quad (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1),$$

and ${}_1F_1(a; b; s) = \sum_{n=0}^{\infty} (a)_n s^n / n! (b)_n$. *F. G. Dressel.*

Thorn, R. J., and Simpson, O. C. Temperature gradients in inductively heated cylinders. J. Appl. Phys. 24, 297–299 (1953).

Let $T(r)$ denote the steady state absolute temperature at a distance r from the axis of a hollow cylinder of infinite length subjected to inductive heating. The cylinder, placed in a vacuum, loses heat by radiation according to the fourth power law at its outer surface. An exact formula is derived for the difference $T(b) - T(a)$ of the temperatures of its inside and outside surfaces in terms of the electromagnetic and thermal coefficients involved. Numerical results based upon this formula are compared with those based on an approximate parabolic distribution of the magnetic intensity.

R. V. Churchill (Ann Arbor, Mich.).

Žautykov, O. A. Cauchy's problem for a denumerable system of partial differential equations. Izvestiya Akad. Nauk Kazah. SSR **1950**, no. 97, Ser. Mat. Meh. **4**, 115–125 (1950). (Russian)

Under suitable continuity hypotheses on the initial Cauchy data and the functions ω_s, the author uses the method of successive approximations in an attempt to prove a local existence and uniqueness theorem for the following Cauchy problem for a denumerable system of third-order equations:

$$\frac{\partial^3 u_s}{\partial x \partial y \partial z} = \omega_s\Big(x, y, z; u_1, u_2, \cdots; \frac{\partial u_1}{\partial x}, \frac{\partial u_1}{\partial y}, \frac{\partial u_1}{\partial z}, \frac{\partial u_2}{\partial x}, \frac{\partial u_2}{\partial y}, \frac{\partial u_2}{\partial z}, \cdots; \frac{\partial^2 u_1}{\partial x \partial y}, \frac{\partial^2 u_1}{\partial x \partial z}, \frac{\partial^2 u_1}{\partial y \partial z}, \frac{\partial^2 u_2}{\partial x \partial y}, \frac{\partial^2 u_2}{\partial x \partial z}, \frac{\partial^2 u_2}{\partial y \partial z}, \cdots \Big),$$

$$u_s(x, y, z)\,|_{x=0} = f_s(y, z), \quad \frac{\partial u_s}{\partial x}(x, y, z)\,|_{x=0} = \varphi_s(y, z),$$

$s=1, 2, \cdots$ (as far as second order derivatives are concerned, only mixed second order derivatives appear as arguments of the functions ω_s). The functions ω_s are supposed to be equicontinuous, equibounded, and to satisfy a certain Cauchy-Lipschitz condition with respect to all arguments save x, y, z. (See the paper reviewed below.)

J. B. Diaz (College Park, Md.).

Žautikov, O. A. Cauchy problems for a denumerable system of partial differential equations of nth order. Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 142–153 (1951). (Russian)

Under suitable continuity hypotheses on the initial Cauchy data and the functions ω_s, the author employs the method of successive approximations in an attempt to prove an (in the small) existence and uniqueness theorem for the following Cauchy problem for a denumerable system of nth order partial differential equations in n independent variables $x_1, \cdots, x_n$:

$$\frac{\partial^n u_s}{\partial x_1 \partial x_2 \cdots \partial x_n} = \omega_s\Big(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots; \frac{\partial u_1}{\partial x_1}, \cdots, \frac{\partial u_1}{\partial x_n}, \frac{\partial u_2}{\partial x_1}, \cdots, \frac{\partial u_2}{\partial x_n}, \cdots; \frac{\partial^2 u_1}{\partial x_1 \partial x_2}, \cdots; \frac{\partial^{n-1} u_1}{\partial x_1 \cdots \partial x_{n-1}}, \frac{\partial^{n-1} u_1}{\partial x_1 \cdots \partial x_{n-2} \partial x_n}, \cdots, \frac{\partial^{n-1} u_2}{\partial x_1 \cdots \partial x_{n-1}}, \frac{\partial^{n-1} u_2}{\partial x_1 \cdots \partial x_{n-2} \partial x_n}, \cdots \Big),$$

$$u_s(x_1, x_2, \cdots, x_n)\,|_{x_1=0} = f_s(x_2, x_3, \cdots, x_n),$$

$$\frac{\partial u_s}{\partial x_1}(x_1, x_2, \cdots, x_n)\,|_{x_1=0} = \varphi_s(x_2, x_3, \cdots, x_n),$$

$s=1, 2, \cdots$ (only mixed second, third, $\cdots$, and $(n-1)$th order derivatives appear among the arguments of the functions ω_s). The functions ω_s are supposed to be equicontinuous, equibounded, and to satisfy a certain Cauchy-Lipschitz condition with respect to all arguments save $x_1, \cdots, x_n$.

Reviewer's note. The author has published previously [same Izvestiya **1949**, no. 60, Ser. Mat. Meh. **3**, 85–90; these Rev. **13**, 560; see also the preceding review] the special cases $n=2$ and $n=3$ of this proposition. Unfortunately, the proposition in question is false. For, if it were true, it would have to hold in particular when all the functions ω_s are identically zero; or, what amounts to the same thing, in the case of one equation for a single function u. It suffices to consider $n=2$. The boundary value problem is then

$$(*) \qquad \frac{\partial^2 u}{\partial x \partial y} = 0, \quad u(0, y) = f(y), \quad \frac{\partial u}{\partial x}(0, y) = \varphi(y),$$

which is a Cauchy problem with Cauchy data prescribed on the characteristic line $x=0$. The corresponding homogeneous Cauchy problem with zero Cauchy data has as solution $u(x, y) = X(x) - X(0)$, where $X(x)$ is arbitrary, save for the requirement $X'(0)=0$, so that if (*) has a solution, it is never unique. Further, if $\varphi(y)$ is not a constant, then (*) has no solution. The common error in the three papers occurs when the given Cauchy problem is replaced by an equivalent system of integral equations, when essential "constants of integration" are inadvertently omitted; for example, in equation (3) of the paper dealing with the case $n=2$).

J. B. Diaz (College Park, Md.).

Thews, Gerhard. Über die mathematische Behandlung physiologischer Diffusionsprozesse in zylinderförmigen Objekten. Acta Bioth. Ser. A. **10**, 105–138 (1 plate) (1953).

The methods of separation of variables and of Laplace transforms are briefly described and applied to various cases: constant rate of consumption with material supplied from the outside, and again with material supplied from inside along a coaxial cylinder, in the time dependent case; and the steady state for a cylinder of finite length with a constant gradient along the boundaries of the outer and inner cylinders. *A. S. Householder* (Oak Ridge, Tenn.).

Difference Equations, Special Functional Equations

✶Gel'fond, A. O. Isčislenie konečnyh raznosteĭ. [The calculus of finite differences.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952. 479 pp. 10.20 rubles.

This book is based on the author's book of the same title [ONTI, Moscow-Leningrad, 1936], which he has revised and supplemented with material on calculus of finite differences for complex variables. All of the material presented has been previously published. Chapter I, after disposing of elementary questions, treats the general interpolation process for a triangular table, devoting considerable attention to representation and approximation problems. (A minor error—formula (83), used in the proof of the theorem on p. 67, is "unfounded"—is corrected in the paper reviewed below.) Chapter II is an extensive treatment of convergence and regularity properties for the Newton series with interpolation points $1, 2, \cdots$; the chapter concludes with a treatment, largely based on previous published work of the author, of general interpolation points. Chapter III is concerned with the general problem of determining an entire function having given elements. A typical problem of this type is: find all entire functions $F(z)$ satisfying $F^{(n)}(n)=0$, $n=1, 2, \cdots$. A number of such problems are treated in detail. There is a discussion of the connection of problems of this type with certain moment problems in the complex domain, and finally applications of the theory of infinite linear differential equations to problems escaping the mo-

ment technique. The fourth chapter is an elementary discussion of summation problems, Bernoulli numbers and polynomials, Euler's formula, etc. Most of Chapter V is concerned with a fairly standard treatment of finite difference equations. The last section of Chapter V, on differential equations of finite order, is based on Gelfond's paper in Trudy Mat. Inst. Steklov. **38**, 42–67 (1951) [these Rev. **13**, 929]. This book is very much in the spirit of the modern Russian school concerned with the so-called constructive theory of functions, approximative methods for the solution of differential equations, and so forth. The book is a valuable collection of results in these directions. The exposition is excellent. *J. M. Danskin* (Washington, D. C.).

Gel'fond, A. O. On an interpolation problem. Doklady Akad. Nauk SSSR (N.S.) **84**, 429–432 (1952). (Russian)

The author observes that there is a gap in his proof of the theorem on p. 67 in the book reviewed above, and corrects the gap, improving the result slightly. The theorem in question is concerned with a triangular interpolation table, and the interpolation series for an entire function with respect to that table; under a simple growth condition on the entire function, if the interpolation table is bounded then the interpolation series converges uniformly to the function in any finite part of the plane. *J. M. Danskin.*

Bajraktarević, Mahmud. Sur certaines suites itérées. C. R. Acad. Sci. Paris **236**, 881–883 (1953).
Bajraktarević, Mahmud. Sur certaines suites itérées. II. C. R. Acad. Sci. Paris **236**, 988–989 (1953).
Bajraktarević, Mahmud. Sur certaines suites itérées. III. C. R. Acad. Sci. Paris **236**, 1125–1127 (1953).

The first note examines the functional equation (i) $f[g(2x)]=g(x)$. Let $f(x)$, defined on $[-a, a]$, satisfy the following conditions: (1) $f(x)$ is non-decreasing, $f(-a)\geqq 0$, $f(x)>x$ $(x<a)$, $f(a)=a$. Let $z=d_0, d_1d_2\cdots$ be the binary expansion for each $z\in I=(0, 2)$, and associate with z the sequence $x_n(z, t)=\epsilon_0 f\{\epsilon_1 f[\cdots(\epsilon_n f(t))\cdots]\}$ $(n=0, 1, \cdots)$ where $\epsilon_n=(1-2d_n)/(1-2d_{n-1})$ and $t\in[-a, a]$. Each sequence $x_n(z, 0)$ has at most two limit points, $\underline{\xi}(z)\leqq\bar{\xi}(z)$. Many results are stated for these, and further properties hold if f is in addition assumed to be continuous and strictly increasing. The relation of property (1) to equation (i) is this: Let $g(z)$ be a decreasing function on $[0, 2]$, continuous on the left [on the right], and such that $g(0)=-g(2)=a>0$, $g(z)+g(2-z)\rightarrow 0$ as $z\uparrow z_1$ $(0<z_1\leqq 2)$, and $g(2z_1)=g(2z_2)$ imply $g(z_1)=g(z_2)$. Then there exists a unique $f(x)$ defined on the set E of values of $g(z)$, satisfying conditions (1), and such that $\bar{\xi}(z)$[or $\underline{\xi}(z)$] (as defined above) is equal to $g(z)$. On the complement of E, $f(x)$ can be taken arbitrarily. Moreover, equation (i) holds $(x\in[0, 1])$.

Note II relates the preceding work to the functional equation (2) $\psi(x+1)=f[\psi(x)]$ $(\psi(0)=0)$. Given $f(x)$, continuous and strictly increasing on $(-\infty, \infty)$, with $f(x)>x$ and $f(x)\rightarrow 0$ as $x\rightarrow-\infty$; by a continuous iterate of f is meant a function $\Theta_y(x)$ of two variables satisfying the following relations: (ii) $\Theta_0(x)=x$, $\Theta_1(x)=f(x)$ on $(-\infty, \infty)$; $\Theta_{y+z}(x)=\Theta_y[\Theta_z(x)]$ for $-\infty<x<\infty$, $y, z, y+z\geqq -1$; $\Theta_y(0)$ is continuous and strictly increasing for $-1\leqq y\leqq 0$. Every Θ satisfying (ii) is continuous and strictly increasing in x and y, and for such a $\Theta=f_y(x)$ there exists a unique solution $\psi(y)$ of (2), continuous and strictly increasing, and such that $f_y(x)=\psi[\psi^{-1}(x)+y]$ $(x\in(-\infty, \infty), y\geqq -1)$. Again: Let $\theta(x)$ be continuous and strictly increasing on $0\leqq x<1$, with $\theta(0)=0$ and $\theta(1-0)=f(0)$. Then every solution ψ of (2), continuous and strictly increasing, has the form (iii) $\psi(x)=f_{[x]}[\theta(x-[x])]$. ($[x]$=greatest integer $\leqq x$.) Equations (2) and (iii) are related to the earlier sequences $x_n(z, 0)$.

Finally, note III extends the results to a system of functional equations $f_n[g_{n+1}(2z)]=g_n(z)$ $(n=0, 1, 2, \cdots)$.
I. M. Sheffer (State College, Pa.).

Green, John W. A note on the solutions of the equation $f'(x)=f(x+a)$. Math. Mag. **26**, 117–120 (1953).

Let $f(x)$ be continuous for all real x and satisfy (1) $f'(x)=f(x+a)$, $a\neq 0$. Then $f(x)\in C_\infty$, and this is essentially all that one can say without further conditions on $f(x)$. This paper considers the effect of extra conditions. Some results obtained are as follows. I. $f(x)$ even [$f(x)$ odd] implies $a=2n\pi+\frac{1}{2}\pi$ and $f=A\cos x$ [$f=A\sin x$], A=constant. II. $a>0$ and $f(x)\geqq 0$ implies $a\leqq e^{-1}$ and $f(x)$ is an entire function such that for $x\geqq x_0$,

$$f(x)\geqq f(x_0)\exp\{(x-x_0)S(a)\},$$

where $S(a)$ is the smaller root of the equation (2) $\log S=aS$. III. If $f(x)$ is an entire function of exponential type, then $f(x)=\sum_{n=1}^{N}c_ne^{S_nx}$ (with an additional term cxe^x if $a=e^{-1}$). Here the $\{S_n\}$ are roots of equation (2). *I. M. Sheffer.*

Kuwagaki, Akira. Sur les fonctions de deux variables satisfaisant une formule d'addition algébrique. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 139–143 (1952).
Kuwagaki, Akira. Sur l'équation fonctionnelle rationnelle de la fonction inconnue de deux variables. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 145–151 (1952).

The first of these papers examines the solutions of an algebraic addition formula, for a function $f(x, u)$, having the form

(a) $P\{f(x+y, u+v), f(x, u), f(x, v), f(y, u), f(y, v)\}=0,$

P a polynomial. Suppose $f(x, u)$ satisfies (a), and is analytic and not infinitely multivalent in the respective variables x, u. Let the coefficient of the highest power of $f(x+y, u+v)$ in (a) be $a\{f(x, u), f(x, v), f(y, u), f(y, v)\}$, and suppose certain auxiliary conditions are fulfilled, of which the following are typical: $P\not\equiv 0$ for $x=y=0$ and for $u=v=0$; $a\{f(x, u), f(0, 0), f(y, u), f(0, 0)\}\not\equiv 0$. Then, using results from the theory of those functions of one variable that possess an algebraic addition theorem, it is shown that $f(x, u)$ can be represented in one or other of the two forms $f(x, u)=A_3[u\cdot A_2^{-1}\{A_1(x)\}]$; $f(x, u)=H\{F(x), G(u)\}$. Here each of the functions A_1, A_2, A_3, F, G (of one variable) is an algebraic function either of x or of e^{cx} (c=constant) or of elliptic functions of x; A_2^{-1} is the inverse function of A_2, and H is an algebraic function.

The second paper deals with that subcase of (a) that can be put in the form

(b) $f(x+y, u+v)=R\{f(x, u), f(x, v), f(y, u), f(y, v)\},$

where R is a rational function. More particularly, it is supposed that (b) can be transformed into

(c) $f(x+y, u+v)=(aU+bV+2c)/(pU+qV+2r),$

where

$$U=f(x, u)f(y, v)+f(x, v)f(y, u),$$
$$V=f(x, u)+f(x, v)+f(y, u)+f(y, v),$$

and a, b, c, p, q, r are constants for which $a:b:c\neq p:q:r$. It is then shown that: (I) equation (c) has a solution $f(x, u)$

that is uniform and continuous in the neighborhood of (0, 0), with $f(x, 0)\neq$constant, $f(0, u)\neq$constant, if and only if the rank of $\begin{pmatrix} b & a-q & p \\ c & b-r & q \end{pmatrix}$ is unity; (II) the solutions under the conditions of (I) are linear fractional functions either of $\lambda x+\mu u$ or of $e^{\lambda x+\mu u}$, where λ, μ are arbitrary constants.
I. M. Sheffer (State College, Pa.).

Pastidès, Nicolas. Sur les équations fonctionnelles du type de Poincaré. Compositio Math. **10**, 168–212 (1952).

The system

$$(1)\quad R_j[\varphi_1(u_1, \cdots, u_n), \cdots, \varphi_n(u_1, \cdots, u_n)] = \varphi_j(s_1u_1, \cdots, s_nu_n),\quad j=1, 2, \cdots, n,$$

where the $\{s_k\}$ are constants, was considered by Poincaré [J. Math. Pures Appl. (4) **6**, 313–365 (1890)] in a particular case, and more generally by Picard [Leçons sur quelques équations fonctionnelles, Gauthier-Villars, Paris, 1928], all under the hypothesis $|s_k|>1$. In the present work it is supposed that each $R_j(x_1, \cdots, x_n)$ is analytic in a neighborhood of the origin, with $R_j(0, \cdots, 0)=0$, and the unknown functions $\varphi_j(u_1, \cdots, u_n)$ are sought as convergent power series around $u_1=\cdots=u_n=0$. Let the linear terms in R_j be $\sum_{k=1}^n a_k{}^j x_k$. A necessary condition for a solution is that $s_1, \cdots, s_n$ be roots of the secular equation $\Phi(s)\equiv|a_i{}^k-s\delta_i{}^k|=0$. By a suitable linear transformation of $\varphi_1, \cdots, \varphi_n$, (1) is carried into a similar system where the linear coefficients now satisfy $a_i{}^i=s_i$, $a_k{}^i=0$ for $i<k$, and $a_k{}^i=0$ for $i>k$ when $s_i\neq s_k$. By an application of the method of dominant functions it is shown that if $s_1, \cdots, s_n$ are the roots of $\Phi(s)$, and if $\Phi(s_1{}^{\mu_1}s_2{}^{\mu_2}\cdots s_n{}^{\mu_n})\neq 0$ for every set $\mu_1, \cdots, \mu_n$ of non-negative integers with sum $\geqq 2$, then (1) has a formal solution in power series for $\varphi_1, \cdots, \varphi_n$; if in addition there exists $\rho>0$ such that $|s^i-s_1{}^{\mu_1}s_2{}^{\mu_2}\cdots s_n{}^{\mu_n}|>\rho$ for all i, then the formal series converge and represent an actual solution.

An extension of system (1) is made to where the number of variables u_k present is $m\neq n$. Then the case of (1) is treated where there exist n integers $p_1, \cdots, p_n$ greater than unity such that $s_1{}^{p_1}=\cdots=s_n{}^{p_n}=1$. Next, system (1) is generalized to

$$\text{(i)}\quad Q_j[F_1(u_1, \cdots, u_n), \cdots, F_n(u_1, \cdots, u_n)] = F_j[S_1(u_1, \cdots, u_n), \cdots, S_n(u_1, \cdots, u_n)],\quad j=1, 2, \cdots, n,$$

where $\{Q_j\}$, $\{S_j\}$ are functions analytic around the origin and vanishing there, and where the secular equations corresponding to these two sets of functions have the same zeros $s_1, \cdots, s_n$; and permutable systems of functions are treated. A final chapter gives an abstract treatment of a mathematical system that includes some of the preceding work on permutable systems. *I. M. Sheffer* (State College, Pa.).

Anastassiadis, Jean. Fonctions semi-monotones et semi-convexes et solutions d'une équation fonctionnelle. Bull. Sci. Math. (2) **76**, 148–160 (1952).

For a fixed real number ω, a function $f(x)$ is said by the author to be semi-monotone ω provided it satisfies one of the relations $f(x+\omega)\geqq f(x)$, $f(x+\omega)\leqq f(x)$, and to be semi-convex ω provided it satisfies $f(x+\omega)\leqq\frac{1}{2}[f(x)+f(x+2\omega)]$, throughout its domain of definition. A. Mayer has shown [Acta Math. **70**, 57–62 (1938)] that a convex function $f(x)$, positive for $x>0$, satisfies the functional relation

$$(1)\qquad f(x+1)=1/xf(x)$$

if and only if

$$(2)\qquad f(x)=G(x)=\frac{1}{\sqrt{2}}\Gamma\left(\frac{x}{2}\right)\Big/\Gamma\left(\frac{x+1}{2}\right).$$

The author now shows that a function $f(x)$, positive for $x>0$, which is semi-decreasing 1, or semi-convex 1, satisfies (1) if and only if $f(x)$ is the function (2). The result is generalized to show that a function $f(x)$, positive for $x>0$, which is semi-decreasing 1, or semi-convex 1, satisfies $f(x+1)=1/P(x)f(x)$, where

$$P(x)=(x+\rho_1)\cdots(x+\rho_k)\quad(\rho_1, \cdots, \rho_k\geqq 0),$$

if and only if $f(x)=G(x+\rho_1)\cdots G(x+\rho_k)$.
E. F. Beckenbach (Los Angeles, Calif.).

Parodi, Maurice. Application de la transformation de Laplace à deux variables, à la résolution d'équations fonctionnelles. C. R. Acad. Sci. Paris **235**, 1597–1598 (1952).

The author uses the Laplace transformation to show that the solution of the functional equation

$$\frac{u^n\varphi(u)-v^n\varphi(v)}{u-v}+\frac{u\varphi(u)-v\varphi(v)}{u-v}=\sum_{k=0}^{n-2}\sum_{m=0}^{k}A_{n-2-k}u^mv^{k-m}$$

is $\varphi(s)=\sum_{k=1}^n\alpha_k/(s-r_k)$, where the r_k are the nth roots of -1 and the α_k are constants determined by the A_k except for one arbitrary constant. *A. Erdélyi* (Pasadena, Calif.).

Integral Equations

★Petrovskij, I. G. Vorlesungen über die Theorie der Integralgleichungen. Übersetzt von R. Herschel. Physica-Verlag, Würzburg, 1953. 100 pp. DM 7.80.

Translated from Petrovskiĭ's Lekcii po teorii integral'nyh uravneniĭ, 2d ed. [Gostehizdat, Moscow-Leningrad, 1951; these Rev. **13**, 467].

Fenyö, I. Über eine Klasse von Integralgleichungen. Publ. Math. Debrecen **2**, 248–251 (1952).

The paper extends the theorem of Egerváry [Math. Phys. Lapok **23**, 303–355 (1914)] that if $K(t)$ is of integrable square and period 2π, then the characteristic values of $K(x-y)$ are $\lambda_n=1/\int_0^{2\pi}K(t)e^{-int}dt$, with corresponding characteristic functions $\varphi_n(t)=Ce^{-int}$, to the case where the points P range over the surface of the unit sphere, and the kernel $K(P, P')$ is dependent only on the angle γ between the radius vectors from 0 to P and P'. It is shown that if, in the expansion of a symmetric $K(\gamma)=f(\cos\gamma)=f(\xi)$ according to Legendre polynomials in ξ, c_n is the coefficient of $P_n(\xi)$, then the characteristic values of $K(P, P')$ are $\lambda_n=(2n+1)/4\pi c_n=(2n+1)/4\pi\int_{-1}^1 f(\xi)P_n(\xi)d\xi$, and the corresponding characteristic functions are the $2n+1$ spherical harmonics of order $n: S_{n,1}(\theta, \varphi), \cdots, S_{n,2n+1}(\theta, \varphi)$.
T. H. Hildebrandt (Ann Arbor, Mich.).

Salam, Abdus. Fredholm solutions of partial integral equations. Proc. Cambridge Philos. Soc. **49**, 213–217 (1953).

The integral equation considered is

$$F(x_1, x_2)=f(x_1, x_2)+\lambda\int K(x_1, x_2, y)(F(y, x_2)+F(x_1, y))dy,$$

where f and K are symmetric in x_1 and x_2 and a symmetric

solution F is sought. The equations are replaced by

$$F(x_1, x_2) = f(x_1, x_2) + \lambda_1 \int K_1(x_1, x_2, y) F(y, x_2) dy + \lambda_2 \int K_2(x_1, x_2, y) F(x_1, y) dy,$$

symbolically, $F = f + \lambda_1 K_1 F + \lambda_2 K_2 F$. Assuming a reciprocal for $I - \lambda_1 K_1$ in (x_1, y) for all x_2 and for $I - \lambda_2 K_2$ in (y, x_2) for all x_1, multiplication of this equation by these reciprocals in either order gives two equivalent equations of the form

$$F(x_1, x_2) = g(x_1, x_2) + \int L(x_1, x_2; y_1, y_2) F(y_1, y_2) dy_1 dy_2,$$

where g and L depend on f, λ_1, λ_2, K_1 and K_2. By proper combination of these equations a symmetric solution of the original equation may be obtained. Extensions to the case of more than two variables are outlined.

T. H. Hildebrandt (Ann Arbor, Mich.).

Gahov, F. D. Concerning a note of I. C. Gohberg. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 181–182 (1952). (Russian)

The author criticises the paper of Gohberg [Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 149–156 (1952); these Rev. **14**, 54 which see for notations], and calls attention to his 1941 dissertation [Izvestiya Kazan. Fiz.-Mat. Obšč. (3) **14**, 75–159 (1949); unavailable for review], to the work of D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 423–452 (1948); **15**, 75–82 (1951); these Rev. **10**, 305; **12**, 832] and the (unavailable) dissertation of L. A. Čikin [Singular cases of Riemann's boundary problem and of singular equations, Kazan, 1952]. He states without proof a result that the singular operator A is regularisable if $A\varphi(t) = g(t)$ is soluble, and this he considers a less restrictive result than Gohberg's that the non-vanishing of $a^2(t) - b^2(t)$ was not only sufficient but necessary when the regularising operator M was to be bounded on L_2 into L_2. On the other hand, Gohberg did not require solubility, a term not precisely defined in the note under review. *F. V. Atkinson.*

Mandžavidze, G. F. On a class of singular integral equations with discontinuous coefficients. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 269–274 (1950). (Russian)

The author makes use of some of the notions and definitions found in Mushelišvili, Singular integral equations ... [OGIZ, Moscow-Leningrad, 1946; these Rev. **8**, 586]. A study is made of the equation

$$(1)\quad K\phi = A(t_0)\phi(t_0) + \frac{1}{\pi i} B(t_0) \int_L \frac{\phi(t)dt}{t - t_0} + \int_L K_1(t_0, t)\phi(t)dt + \int_L \bar{K}_2(t_0, t)\bar{\phi}(t)dt = f(t_0),$$

the coefficients and f being in H (Hölder class) on L, except at most for a finite number of discontinuities ξ of the first kind; the unknown $\phi(t)$ is to be in H, except perhaps at the ξ, where $\phi(t)$ may have discontinuities of order <1; L is a finite set of disjoint, simple, closed, suitably smooth curves, bounding a connected domain. The adjoint of $K(\phi)$ is

$$K'\psi = A(t_0)\psi(t_0) - \frac{1}{\pi i} \int_L \frac{B(t)\psi(t)}{t - t_0} dt + \int_L \psi(t) K_1(t, t_0) dt + \int_L \bar{\psi}(t) \bar{K}_2(t, t_0) dt.$$

The author finds all the solutions of (1) in the class h. Of importance are the canonic function $Z(t)$ and the index H of (1). Some of the results are as follows. In order that $K\phi = f$ could be solved in h it is necessary and sufficient that $\Re\{\int_L f\psi_j dt\} = 0$ $(j = 1, \cdots, k')$, where the $\psi_j(t)$ constitute a complete system of linearly independent solutions in the class h′ of $K'\psi = 0$. If k is the number of linearly independent solutions in h of $K\phi = 0$ and k' is the number of linearly independent solutions in h′ of $K'\psi = 0$, then $k - k' = 2H$, where H is the index of the class h of $K\phi = 0$.

W. J. Trjitzinsky (Urbana, Ill.).

Mandžavidze, G. F. On a system of singular integral equations with discontinuous coefficients. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 351–356 (1950). (Russian)

The author presents a natural extension to systems of his work relating to a single equation, as indicated in the preceding review. With the aid of the generalized Noether theorems for a certain matrix integral equation, associated with the given system, the author establishes the fundamental theorems for his system. The results are similar to those obtained in the case of a single equation.

W. J. Trjitzinsky (Urbana, Ill.).

Mihlin, S. G. Concerning a theorem on boundedness of a singular integral operator. Uspehi Matem. Nauk (N.S.) **8**, no. 1(53), 213–217 (1953). (Russian)

In a previous expository work [1] [Uspehi Matem. Nauk (N.S.) **3**, 3(25), 29–112 (1948); these Rev. **10**, 305] the author made use of his earlier theorem, according to which a singular integral operator is bounded in L_2 if its symbol is bounded. The author renders greater precision to this theorem, important in view of some of its consequences. The notation is that from [1]. If the symbol of the simplest singular operator depends only on θ and is bounded, then the norm of the operator in $L_2(E_m)$ does not exceed the maximum of the modulus of the operator. Whence, if the symbol of the simplest singular operator is of the form $\sum a_n(M_0)\Phi_n(\theta)$ and the series $s = \sum \max|a_n(M_0)| \cdot \max|\Phi_n(\theta)|$ converges, then the operator is bounded in $L_2(E_m)$ and its norm is $\leq s$. The operator (2) $Au = \int_{E_m} u(M_1) r^{-m} f(M_0, \theta) dr_1$ is bounded in $L_2(E_2)$ if the characteristic $f(M_0, \theta)$ has derivatives with respect to θ of order $\leq 2m - 3$, continuous in θ and bounded independently of M_0, and

$$\int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} |\partial^{2m-2} f / \partial\vartheta_1^2 \cdots \partial\vartheta_{m-1}^2|^2 d\vartheta_1 \cdots d\vartheta_{m-1} \leq C \text{ (constant)}.$$

The two-dimensional singular integral operator is bounded in $L_2(E_2)$ if its characteristic satisfies

$$\int_0^{2\pi} |f''(M_0, \vartheta)| d\vartheta \leq C \text{ (constant)}.$$

W. J. Trjitzinsky (Urbana, Ill.).

Karcivadze, I. N., and Hvedelidze, B. V. On an inversion formula. Soobščeniya Akad. Nauk Gruzin. SSR **10**, 587–591 (1949). (Russian)

Let C_k $(k = 1, 2, \cdots)$ be simple closed, suitably smooth curves, exterior to each other and with a positive direction assigned on each; $C = \sum C_k$; it is assumed that no point of accumulation of the sequence $C_1, C_2, \cdots$ belongs to C; C_k is given by $t = t(s)$, $0 \leq s \leq l_k$, where s is length of arc and l_k is the length of C_k. Let $\omega(\tau; \phi) = \sup(|t_2 - t_1| \leq \tau) |\phi(t_2) - \phi(t_1)|$

for $0<\tau\leqq l_k$; I_p is the class of $\phi(t)$ for which

$$\int_0^{l_k}\omega(\tau;\phi)\log^p\frac{l_k}{\tau}\frac{d\tau}{\tau}$$

is finite; $I_\infty=\prod I_p$ $(p\geqq 0)$. Such classes have been previously studied by L. G. Magnaradze [same Soobščeniya **8**, 509–516, 585–590 (1947); these Rev. **14**, 152]. It is said that ϕ belongs to A on C if ϕ is in I_∞ on C_k for $k=1, 2, \cdots$ and if the series

$$\text{(1)}\qquad \pi i\Phi(t)\equiv\int_C\phi(\zeta)\frac{d\zeta}{\zeta-t}=\sum_k\int_{C_k}\cdots$$

converges uniformly on each constituent curve of C (but not necessarily on C). It is proved that, if ϕ is in A, then $\pi i\phi(x)=\int_C\Phi(t)(t-x)^{-1}dt$ for x on C. The integral equation

$$\text{(2)}\qquad a\phi(t)+\frac{b}{\pi i}\int_C\phi(\zeta)\frac{d\zeta}{\zeta-t}=f(t)$$

[f in A on C, constants a and b, with $a^2\neq b^2$] is solved by an explicit formula in the class A. The function ϕ belongs to A_p [A_p^*] if ϕ is in I_p [I_{p-1}] on each C_k and if (1) converges uniformly on each curve and represents thereon a function of class I_{p-1} [I_p]. The equation (2), where f is in A_p, has a unique solution in A_p^* $(p\geqq 2)$. *W. J. Trjitzinsky.*

Karcivadze, I. N. On the behavior of an integral of Cauchy type near the ends of the path of integration. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 256–263 (1951). (Russian. Georgian summary)

The author studies the Cauchy integral

$$\text{(1)}\qquad \Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(t)}{t-z}dt$$

near an end point c of the open curve L. If (i) $\phi(t)$ is in H on every arc interior to L and if (ii) near c one has $\phi(t)=(t-c)^{-\alpha}\phi^*(t)$ $(0\leqq\Re\{\alpha\}<1)$, $\phi^*(t)$ being in H near c, c included, then, according to Muskhelišvili [Singular integral equations . . . , OGIZ, Moscow-Leningrad, 1946; these Rev. **8**, 586] $(z-c)^\alpha\Phi(z)$ is bounded near $z=c$; here $(z-c)^\alpha$ is a suitable branch, single-valued in the plane, cut along L. The author investigates (1) when the condition (ii) is deleted. When L is smooth and ϕ is Lebesgue integrable over L, then $(z-c)^\alpha\Phi(z)$ is bounded near c outside any closed angular neighborhood of c, which contains a portion of L abutting on c. The situation generally becomes quite complicated. The modulus of $\Phi(z)$ may be made to increase arbitrarily fast, when z tends to c, provided that the density of the integral (1) and the mode of approach to c are suitably chosen. *W. J. Trjitzinsky* (Urbana, Ill.).

Vekua, N. P. Hilbert's boundary problem for several unknown functions in the case of unconnected regions. Soobščeniya Akad. Nauk Gruzin. SSR **11**, 533–538 (1950). (Russian)

The author studies the boundary-value problem of Hilbert for several unknowns and, correspondingly, a system of singular integral equations in the case of disjoint domains, bounded by closed, smooth contours, which may intersect themselves and each other in a finite number of points. The case of a single unknown has been studied in at least two previous works [Trjitzinsky, Trans. Amer. Math. Soc. **60**, 167–214 (1946); these Rev. **8**, 211; Kveselava, Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 1–27 (1949); these Rev. **13**, 135]; in the first of these, open curves were admitted. The author points out the interesting fact that transition from one to several unknowns cannot be made directly. Let $L_1, \cdots, L_m$ be closed, smooth contours; L_i contains L_{i-1} $(i=2, \cdots, m)$; $L=L_1+\cdots+L_m$; L_1 bounds a domain D_1^+, and the domains enclosed between the contours L_{2k}, L_{2k+1} and L_{2k-1}, L_{2k} are respectively denoted by D^+_{2k+1}, D^-_{2k}; the infinite domain exterior to L_m is D^+_{m+1} if m is even, and is D^-_{m+1} if m is odd. The positive direction on L leaves D_k^+ on the left. The Hilbert problem studied is as follows. To find a piecewise analytic vector $\Phi(z)=(\Phi_1, \cdots, \Phi_n)$ of finite order at ∞, so that, on L, $\Phi_\alpha^+=G_{\alpha 1}\Phi_1^-+\cdots+G_{\alpha n}\Phi_n^-+g_\alpha$ $(\alpha=1, \cdots, n)$, where the $G_{\alpha\beta}$, g_α are in H on L; in the matrix form: $\Phi^+=G\Phi^-+g$. It is assumed that the determinant of the matrix $G=(G_{\alpha\beta})$ does not vanish on L. The author first gives an effective solution of the homogeneous problem $(g\equiv 0)$. Use is made in succession of canonical matrices of suitable boundary problems for the curves $L_1, L_2, \cdots$; finally, a certain matrix $\chi(z)=(\overset{k}{\zeta}_j(z))$ $(j, k=1, \cdots, n)$ is obtained so that $\chi^+=G\chi^-$ on L. The vectors $\overset{k}{\chi}=(\overset{k}{\chi}_1, \cdots, \overset{k}{\chi}_n)$ constitute a system of solutions such that: (1) $|\chi(z)|\neq 0$ for z finite; (2) on letting $\overset{k}{\chi}{}^0(z)=z^{H_k}\overset{k}{\chi}(z)$ ($-H_k$ being the order of the solution $\overset{k}{\chi}(z)$ at ∞), the determinant $|\overset{k}{\chi}{}_j^0(\infty)|\neq 0$, $\neq\infty$. The system χ is termed canonical; the H_j are partial indices; $H=H_1+\cdots+H_n$ is the total index. All the solutions $\Phi(z)$ of $\Phi^+=G\Phi^-$ are given by $\Phi=\chi p$; where the matrix χ is canonical and p is a polynomial vector. Next are solved the nonhomogeneous boundary problem $\Phi^+=G\Phi^-+g$ and, correspondingly, the singular integral matrix equation $A(t_0)\rho(t_0)+(\pi i)^{-1}\int_L K(t_0, t)\rho(t)(t-t_0)^{-1}dt=f(t_0)$ (in the class H). The author indicates how the above theory continues to apply, when a finite number of open curves is admitted.

W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. P. On a problem of Hilbert with discontinuous coefficients and its application to singular integral equations. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 307–313 (1951). (Russian. Georgian summary)

Let L be a regular closed contour, bounding a connected domain D^+ in the plane ω; $D^-=\omega-(D^++L)$; $\phi(z)$ is piecewise analytic if $\phi(z)$ is analytic in D^+ and in D^- (for $z\neq\infty$) and if ϕ can be continuously extended on L from either side, except perhaps for a finite number of points c, in the vicinity of which $\phi(z)=O(|z-c|^{-\alpha})$ $(0\leqq\alpha<1)$. The problem considered is to find a piecewise analytic ϕ, of finite order at ∞, so that (1) $\phi^+=G_0\phi^-+g_0$. When G_0, g_0 are in H on L and G_0 is nowhere zero on L, the complete solution of (1) is already known; the same can be said of a corresponding type (2) of singular integral equations. The author solves the problem (1) and the equation (2) when G_0 (and a certain appropriate function in the case of (2)) is allowed to have at some points of L discontinuities of order <1 or zeros of order <1. These developments naturally involve canonic solutions and the use of the index.

W. J. Trjitzinsky (Urbana, Ill.).

Ganin, M. P. Equivalent regularization of systems of singular integral equations. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 517–523 (1951). (Russian)

The author studies the system

$$\text{(1)}\quad K\phi\equiv A(t)\phi(t)+\frac{B(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau+\int_L K(t,\tau)\phi(\tau)d\tau=g(t),$$

where A, B, K are matrices, g, ϕ (unknown) are vectors in the class H on L; $K(t,\tau)=|t-\tau|^{-\lambda}K_0(t,\tau)$ $(0\leqq\lambda<1)$, where the matrix $K_0(t,\tau)$ is in H in both variables; L is a finite sum of disjoint, simple, closed, smooth curves limiting a domain S^+. It is said that R is an equivalently regularizing operator for (1) if R transforms (1) into an equivalent system of Fredholm integral equations, each solution of one system being a solution of the other. The author solves the problem of equivalence for (1) completely, using suitable generalizations (extensions) of methods due to Kupradze [Boundary problems in the theory of vibrations . . . , Moscow-Leningrad, 1950] and Vekua [Soobščeniya Akad. Nauk Gruzin. SSR **3**, 869–876 (1942); these Rev. **5**, 268].

W. J. Trjitzinsky (Urbana, Ill.).

Ganin, M. P. On a generalized system of singular integral equations. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 591–596 (1951). (Russian)

Let S^+ be a finite domain of connectivity $m+1$, bounded by disjoint regular, closed contours $L_0, L_1, \cdots, L_m$, the first of these containing the rest; $d(t)$ transforms $L=L_0+\cdots+L_m$ one-to-one on itself; t and $\alpha(t)$ describe L in the same direction; $\beta(t)$ is the inverse of $\alpha(t)$; $\alpha^{(1)}(t)$, $\beta^{(1)}(t)$ are nowhere zero on L and belong to the class H on L. Considered is the system

$$(1)\quad K\phi(t)\equiv A(t)\phi[\alpha(t)]+B(t)\phi(t)+\frac{A(t)}{\pi i}\int_L\frac{\phi(\tau)}{\tau-\alpha(t)}d\tau$$
$$-\frac{B(t)}{\pi i}\int_L\frac{\phi(\tau)}{\tau-t}d\tau+\int_L K(t,\tau)\phi(\tau)d\tau=g(t);$$

matrices A, B and vector g are assigned in H on L; matrix $K(t,\tau)=|\tau-t|^{-\lambda}K_1(t,\tau)+|\tau-\alpha(t)|^{-\mu}K_2(t,\tau)$ $(0\leqq\lambda,\mu<1)$, where matrices K_ν $(\nu=1,2)$ are assigned in H in both variables on L; det $A\neq0$, det $B\neq0$ on L; vector ϕ is to be found in H. The author simplifies and completes the earlier solution of this problem, as given by N. P. Vekua [Systems of singular integral equations . . . , Gostehizdat, Moscow-Leningrad, 1950; these Rev. **13**, 247]. The notable advance is that with respect to equivalence. It is shown that the system (1) is equivalent to an ordinary system of singular integral equations; also, a new proof is given of the generalized Noether theorems.

W. J. Trjitzinsky.

Harazov, D. F. Application of integral equations with kernels depending upon a parameter to some boundary problems in the theory of differential equations. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 265–306 (1951). (Russian. Georgian summary)

The author shows that a number of important self-adjoint ordinary differential and partial differential boundary problems (D) $\{L(u)=Q_m(x;\lambda)u-f(x)$ plus a boundary condition$\}$ are equivalent to an integral equation of the form

$$(1)\quad u(x)=\int_T G(x,y)Q_m(y;\lambda)u(y)dy-\int_T G(x,y)f(y)dy,$$

where $Q_m(y;\lambda)=r_1(y)\lambda+\cdots+r_m(y)\lambda^m$. Here $\phi(x)=\phi(x_1,x_2)$ in the case of a plane; double integration over a domain T is denoted by $\int_T$ and a quadruple integral over a domain T^2 of the 4-space $((x_1,x_2,y_1,y_2)=(x,y)\in T^2$ if $x\in T$ and $y\in T)$ by $\int\int_{T^2}$; T is a segment $[a,b]$ when (1) arises from an ordinary differential equation. Domains T are assumed to possess suitably regular boundaries; the coefficients involved are continuously differentiable up to a certain order (and some of them satisfy Hölder conditions) in the closure of T. Here $G(x,y)$ is an appropriate Green's function. The conditions are such that the kernel is of integrable square in T^2; it is symmetric, positive-definite. The homogeneous problems (D) lead to $(1;f=0)$. The spectra of homogeneous problems (D) coincide with the spectra of the corresponding problems $(1;f=0)$, and the study of problems (D) is reduced to the study of characteristic functions of $(1;f=0)$ and the solutions of (1). This is a quite substantial piece of work, containing many results, which space does not permit to state here. It will suffice to say that the motif of the work is as indicated above; on the other hand, the hypotheses are such that the integral equation theory, brought into play, is essentially on the level of the classical Hilbert-Schmitt theory.

W. J. Trjitzinsky (Urbana, Ill.).

Aleksandriya, G. N. Generalized problem of Haseman for several unknown functions. Soobščeniya Akad. Nauk Gruzin. SSR **12**, 585–590 (1951). (Russian)

Let L be a simple, closed, suitably smooth curve, limiting a finite domain S^+; S^- is the complement of S^++L; $\alpha(t)$ transforms L one-to-one on itself; the derivative $\alpha'(t)$ is in H; t and $\alpha(t)$ describe L in opposite directions; a function is meromorphic in S^+ if it is analytic in S^+ except for a finite number of poles and if it can be continuously extended to L. The problem considered is to find in S^+ meromorphic vectors $\phi_\nu=[\phi_{\nu1},\cdots,\phi_{\nu n}]$ $(\nu=1,2)$ so that on L

$$(1)\qquad \phi_1^+[\alpha(t_0)]=G(t_0)\overline{\phi_2^+(t_0)}+g(t_0),$$

where the matrix $G=(G_{kj})$ $(k,j=1,\cdots,n)$ and the vector g are assigned in H on L; det $G\neq0$ on L. The homogeneous problem $(1;g=0)$ is transformed into a system of singular integral equations of normal type and of index zero. If the adjoint problem of $(1;g=0)$ has no analytic solutions, then $(1;g=0)$ has meromorphic solutions (given explicitly and involving integrations in the sense of principal values and standard rational vectors). The problem $(1;g=0)$ is always solvable in terms of certain singular integrals, the expressions for their densities involving a solution of a certain singular integral equation. The author obtains all the solutions of $(1;g=0)$, having at the origin singularities of order $\geqq-(r+1)$. It is indicated that the above can lead to the proof of existence of canonical solutions, in terms of which all the solutions of $(1;g=0)$ and of (1) are representable.

W. J. Trjitzinsky (Urbana, Ill.).

Bellman, Richard, and Latter, Richard. On the integral equation $\lambda f(x)=\int_0^a K(x-y)f(y)dy$. Proc. Amer. Math. Soc. **3**, 884–891 (1952).

$K(x)$ is assumed to be even, non-negative and monotone decreasing for $0\leqq x<\infty$. The boundary a is taken positive; the integral $\int_0^\infty K(x)dx=c$ shall exist. The main results are bounds for the largest characteristic value λ, as given by $2\int_0^{a/2}K(x)dx\geqq\lambda\geqq2\int_0^aK(x)dx-2a^{-1}\int_0^axK(x)dx$. From this it follows that $\lambda\rightarrow2c$ for $a\rightarrow\infty$. Two proofs are given, both using the fact that $f(x)$ can be chosen as positive. The first one uses the well-known variational formula

$$(1)\qquad \lambda\int_0^a g^2(x)dx\geqq\int_0^a\int_0^a K(x-y)g(x)g(y)dxdy$$

in combination with a variational principle of Bohnenblust. The second proof uses $f(x)=f(a-x)$ (which is trivial) and the property that $f(x)$ is monotone increasing for $0\leqq x\leqq a/2$. The reviewer believes that the results can be obtained more easily. The lower bound for λ follows from (1) for $g(x)\equiv1$ by straightforward calculation. The upper bound results from

integrating the equation in the title. This gives

$$\lambda\int_0^a f(x)dx=\int_0^a K(x-y)f(y)dxdy$$
$$\leqq \operatorname{Max}_y \int_0^a K(x-y)dx\cdot\int_0^a f(y)dy$$
$$\leqq 2\int_0^{a/2} K(x)dx\cdot\int_0^a f(y)dy.$$

H. Bückner (Minden).

Huang, Su-Shu. The variational method for problems of radiative transfer. I. Isotropic scattering with a constant net flux. Astrophys. J. 117, 211–214 (1953).

Expressing the solution of the Schwarzschild-Milne integral equation

$$(1)\qquad J(\tau)=\frac{1}{2}\int_0^\infty J(t)E_1(|t-\tau|)dt$$

(where E_n denotes the exponential integral of order n) in the form $J(\tau)=\tau+q(\tau)$, the author solves the nonhomogeneous integral equation for q,

$$(2)\qquad q(\tau)=\frac{1}{2}\int_0^\infty q(t)E_1(|t-\tau|)dt+\tfrac{1}{2}E_3(\tau),$$

by a variational principle. This latter principle depends on the circumstance [G. Placzek and W. Seidel, Physical Rev. (2) 72, 550–555 (1947); these Rev. 9, 147] that solving (2) is equivalent to minimizing

$$\mathfrak{F}(q)=\int_0^\infty q(\tau)\left\{q(\tau)-\frac{1}{2}\int_0^\infty q(t)E_1(|t-\tau|)dt\right\}d\tau$$
$$-\int_0^\infty q(\tau)E_3(\tau)d\tau.$$

Adopting for q a trial function of the form

$$q(\tau)=A_1+\sum_{n=2}^{5}A_nE_n(\tau)$$

where $A_1, \cdots, A_5$ are variational parameters, the author determines the A's by minimizing $\mathfrak{F}(q)$ evaluated with the chosen form of q. The solution obtained in this manner is found to be in good numerical agreement with those obtained by other methods. *S. Chandrasekhar.*

Huang, Su-Shu. The variational method for problems of radiative transfer. II. The formation of absorption lines in the Milne-Eddington model. Astrophys. J. 117, 215–220 (1953).

The so-called Milne-Eddington model for the formation of absorption lines in stellar atmospheres leads to the integral equation

$$(1)\qquad J(\tau)=(1-\lambda)(c_0+c_1\tau)+\tfrac{1}{2}\lambda\int_0^\infty J(t)E_1(|t-\tau|)dt,$$

where λ, c_0 and c_1 are constants. Writing the solution of (1) in the form $J(\tau)=c_0+c_1t+q(\tau)$, the author observes that solving the integral equation,

$$q(\tau)=\tfrac{1}{2}\lambda\int_0^\infty q(t)E_1(|t-\tau|)dt+\tfrac{1}{2}\lambda[c_1E_3(\tau)-c_0E_2(\tau)],$$

governing q is equivalent to minimizing

$$\mathfrak{F}(q)=\int_0^\infty q(\tau)\left\{q(\tau)-\tfrac{1}{2}\lambda\int_0^\infty q(t)E_1(|t-\tau|)dt\right\}d\tau$$
$$-\lambda\int_0^\infty q(\tau)[c_1E_3(\tau)-c_0E_2(\tau)]d\tau.$$

Adopting for q the trial functions

$$q(\tau)=A_1e^{-m\tau}+A_2E_2(\tau)+A_3E_3(\tau),$$

where m is the positive real root of the transcendental equation $m=\lambda\tanh^{-1}m$ and the A's are the variational parameters, the author determines these latter by minimizing $\mathfrak{F}(q)$ evaluated with the chosen trial function. The solution obtained in this manner is found to be in agreement with those obtained by other methods. *S. Chandrasekhar.*

Huang, Su-Shu. The variational method for problems of radiative transfer. III. Reflection effect. Astrophys. J. 117, 221–224 (1953).

The integral equations governing the problem of the diffuse reflection by an isotropically scattering semi-infinite atmosphere is

$$J(\tau)=\frac{1}{2}\int_0^\infty J(t)E_1(|t-\tau|)dt+\tfrac{1}{4}Fe^{-\tau/\mu_0}$$

where F and $\mu_0(<1)$ are assigned constants. The author solves this equation by a variational method similar to the ones described in the two preceding reviews.

S. Chandrasekhar (Williams Bay, Wis.).

Guy, Roland. Sur les solutions de l'équation d'évolution. C. R. Acad. Sci. Paris 235, 1194–1196 (1952).

The equation of evolution is shown to have solutions when the operator $\mathfrak{F}$ is linear and weakly continuous [cf. same C. R. 233, 288–290 (1951); 234, 918–920 (1952); these Rev. 13, 194, 951]. *C. C. Torrance* (Monterey, Calif.).

Germay, R. H. Application de la méthode des fonctions majorantes à l'étude de certaines équations intégro-différentielles récurrentes. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 125–130 (1952).

The author continues his investigations begun in previous works [these Rev. 11, 368; 13, 40]. In this paper the right members of the integro-differential equations are holomorphic functions of their arguments and the author obtains sharper results by the use of majorizing functions.

I. A. Barnett (Cincinnati, Ohio).

Germay, R. H. Application de la méthode des fonctions majorantes à l'étude de certains systèmes d'équations intégro-différentielles récurrentes. II. Ann. Soc. Sci. Bruxelles. Sér. I. 67, 13–18 (1953).

The method of majorizing functions employed in the paper reviewed above to establish the existence of solutions for a single equation is now used for systems of such equations. *I. A. Barnett* (Cincinnati, Ohio).

Lehmann, N. Joachim. Bemerkungen zu einer Klasse polarer Integrodifferentialgleichungen. Math. Nachr. 9, 45–50 (1953).

The present paper is a continuation of earlier work of the author [Math. Nachr. 5, 139–160 (1951); these Rev. 13, 247] on linear integrodifferential equations of the form $y(x)=\lambda(K(x,s),y(s))+f(x)$, when on a suitable linear

function space $\mathfrak{D}$ the inner product

$$(u,v)=\sum_{\nu,\rho=0}^{n}\left\{\int_a^b u^{(\nu)}(x)v^{(\rho)}(x)d\sigma_{\nu\rho}(x)+\sum_{i,k}a_{\nu\rho}^{ik}u^{(\nu)}(x_i)v^{(\rho)}(x_k)\right\}$$

is symmetric, where $\sigma_{\nu\rho}(s)$ are real-valued functions of bounded variation on $a\leqq s\leqq b$, $\{x_i\}$ is a finite sequence of points on $a\leqq x\leqq b$, and $a_{\nu\rho}^{ik}$ are real constants. Theorems on the existence of proper values of $y(x)=\lambda(K(x,s),y(s))$ and associated expansion theorems are herein established in case: (i) $K(x,s)$ is real-valued, symmetric, and such that $K^{\nu,\rho}(x,s)=\partial^{\nu+\rho}K(x,s)/(\partial x^\nu\partial s^\rho)$ $(\nu,\rho=0,1,\cdots,n)$, are continuous in (x,s) on $a\leqq x$, $s\leqq b$, and (ii) on $\mathfrak{D}$ the symmetric bilinear form $\langle u,v\rangle\equiv(u(x),(K(x,s),v(s)))$ satisfies the positive semi-definiteness condition $\langle u,u\rangle\geqq 0$. *W. T. Reid.*

Stapan [Stapans], A. Ê. On nonlinear integral equations with a discrete spectrum of characteristic values. Latvijas PSR Zinātnu Akad. Vēstis **1951**, no. 4 (45), 643–650 (1951). (Russian. Latvian summary)

Let $K(x,y)$ be a continuous symmetric real-valued kernel whose bilinear expansion

$$K(x,y)=\sum_{\nu=1}^{\infty}\frac{\varphi_\nu(x)\varphi_\nu(y)}{\lambda_\nu}$$

in terms of its characteristic functions and characteristic values is uniformly convergent. If the orthonormal system $\{\varphi_\nu(y)\}$ satisfies the additional condition

$$\int_a^b\varphi_{\nu_1}(y)\varphi_{\nu_2}(y)\cdots\varphi_{\nu_{2k+1}}(y)dy=0\quad(1\leqq k\leqq m)$$

for all positive integral values of $\nu_1,\nu_2,\cdots,\nu_{2k+1}$, then the non-linear integral equation

$$u(x)=\lambda\int_a^b K(x,y)\left[u(y)+\sum_{k=1}^{m}C_k u^{2k}(y)\right]dy$$

has a discrete spectrum, identical with that of the linear equation

$$u(x)=\lambda\int_a^b K(x,y)u(y)dy.$$

As an application, it is shown that the boundary value problem

$$y''+\lambda\left[y+\sum_{k=1}^{m}C_k y^{2k}\right]=0,\quad y(0)=y(2\pi)=0,$$

where the C_k are arbitrary constants, has a discrete spectrum. *F. Smithies* (Cambridge, England).

Functional Analysis, Ergodic Theory

Roberts, G. T. Bounded-weak topologies and completeness in vector spaces. Proc. Cambridge Philos. Soc. **49**, 183–189 (1953).

Après avoir rappelé quelques définitions générales, l'auteur donne d'abord une nouvelle démonstration d'un théorème de A. Grothendieck [C. R. Acad. Sci. Paris **230**, 605–606 (1950); ces Rev. **12**, 715] caractérisant le complété du dual d'un espace localement convexe. Généralisant une notion introduite par G. Köthe, une partie S du dual E' d'un espace localement convexe E est dite limitée si, pour tout filtre $\mathfrak{F}$ sur E, contenant un ensemble borné et convergeant faiblement vers 0, $\mathfrak{F}$ est plus fin que le filtre des ϵS^0 (ϵ tendant vers 0). L'auteur montre que cette définition équivaut à dire que S est fortement précompact, et que la topologie définie sur E par les polaires des ensembles limités de E' est la plus fine induisant sur chaque partie bornée de E la topologie faible. La dernière partie du travail énonce plusieurs résultats, d'où il résulterait entre autres que tout dual fortement quasi-complet serait complet et par suite que le dual fort d'un espace tonnelé serait toujours complet. Bien que ne connaissant aucun contre-exemple à ces propositions, le rapporteur estime que la démonstration qu'en donne l'auteur n'est pas convaincante. Le point faible est l'affirmation faite dans le corollaire du théorème 3, et utilisée de façon essentielle par la suite, selon laquelle "the continuous functionals are determined by the convergent sequences". Si, comme le rapporteur le suppose, l'auteur veut dire par là que toute forme linéaire transformant les suites convergentes en suites convergentes est continue, la proposition est inexacte, comme le montrent des exemples bien connus; pour que la démonstration puisse être considérée comme suffisante, il faudrait que l'auteur justifie que, dans le cas qu'il envisage, la proposition citée ci-dessus est vraie. *J. Dieudonné* (Evanston, Ill.).

Shimoda, Isae. Notes on general analysis. I. J. Gakugei Coll. Tokushima Univ. **2**, 13–20 (1952).

This paper is in part a sequel to an earlier paper by the same author [J. Sci. Gakugei Fac. Tokushima Univ. **1**, 1–5 (1950); these Rev. **13**, 356]. In this earlier paper some theorems were proved about the behavior, on the boundary of the sphere of analyticity, of a power series (of homogeneous polynomials) with variable and values in complex Banach spaces. This paper makes a slight extension of the earlier work. It also contains theorems about the radius of analyticity of the series obtained by term-by-term Fréchet differentiation, and about the radius of analyticity of the power series development of $f(x)$ about x_0 if x_0 is inside the sphere of analyticity with center at 0. The last section of the paper deals with the function $f(x,y)$, where x, y and the function values all lie in complex Banach spaces. A direct proof is given that if f is analytic in each variable separately, and bounded on each compact subset of its domain of definition, then f is analytic in x and y jointly. This generalizes a theorem of Osgood. It leads to an alternative proof of the generalized Hartogs theorem [see M. A. Zorn, Ann. of Math. (2) **46**, 585–593 (1945), esp. p. 593; these Rev. **7**, 251]. *A. E. Taylor* (Los Angeles, Calif.).

Kantorovich, L. V. Functional analysis and applied mathematics. Translated by C. D. Benster. U. S. Department of Commerce, National Bureau of Standards, Los Angeles, Calif., NBS Rep. **1509**. ii+202 pp. (1952).

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948); these Rev. **10**, 380.

Kato, Tosio. Notes on some inequalities for linear operators. Math. Ann. **125**, 208–212 (1952).

The author finds new proofs and generalizations of some results given by E. Heinz [Math. Ann. **123**, 415–438 (1951); these Rev. **13**, 471]. The following definition is introduced. If S, T are linear operators in Hilbert space, then $S\ll T$ is equivalent to $\mathfrak{D}_T\supseteq\mathfrak{D}_S$ and $\|Sx\|\leqq\|Tx\|$ for $x\in\mathfrak{D}_T$. Then the theorems proved are: If $S\ll T$ and S^*, T^*, S^{-1} exist, then T^{-1} exists, $\mathfrak{R}_{T^*}\supseteq\mathfrak{R}_{S^*}$, $S^{*\prime}\gg T^{*\prime}$ where by a "$'$" we designate a right inverse. If A, B are self-adjoint, $A\ll B$, A^{-1} exists, then B^{-1} exists and $B^{-1}\ll A^{-1}$. If A, B are posi-

tive and $A \ll B$, then $A^\nu \ll B^\nu$ for $0 \leq \nu \leq 1$. If Q is closed with dense domain, A, B positive, $Q \ll B$, $Q^* \ll A$, then $|(Qx, y)| \leq \|B^\nu x\| \|A^{1-\nu} y\|$ for $x \in \mathfrak{D}_B$, $y \in \mathfrak{D}_A$, $0 \leq \nu \leq 1$.
František Wolf (Berkeley, Calif.).

Halmos, Paul R., Lumer, Günter, and Schäffer, Juan J. Square roots of operators. Proc. Amer. Math. Soc. **4**, 142–149 (1953).

Let H be a complex, infinite-dimensional, Hilbert space. The paper presents a class of operators A on H (i.e., bounded, linear transformations of H into itself) such that each A is invertible but has no square root.
S. Sherman (Sherman Oaks, Calif.).

Monna, A. F. Sur une classe d'espaces linéaires normés. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 513–525 (1952).

Let K be a field complete with respect to a non-trivial discrete valuation, let the residue field have a finite number q of elements, let S be a system of representatives in K of these elements with $0 \in S$, let $a \rightarrow a'$ be a $1-1$ mapping of S onto the set of integers $0, 1, \cdots, q-1$ such that $0'=0$. Let E be a totally non-archimedean normed linear space over K. Assume that there exists a sequence $\{x_n\}$ in E, $-\infty < n < \infty$, such that every element $x \in E$ can be expanded in the form $x = \sum_{-\infty}^{\infty} a_i x_i$, $a_i \in S$, with $a_i = 0$ for almost all negative i. (For significance of this assumption see a previous paper by the author [same Proc. **52**, 151–160 = Indagationes Math. **11**, 40–49 (1949); these Rev. **10**, 549], and other references given there.) Let E^* be the set of all x in E for which the above expansion is not finite, and define $T(x) = \sum_{-\infty}^{\infty} a_i' q^{-i}$. Then T is $1-1$ from E^* to the set R_+ of positive real numbers. T is continuous, and so is T^{-1} except at the q-adic rationals, at which points it is still left continuous. If a measure is given on R_+, it can be carried over to E^* and thence to E. This transfer is investigated in detail. The case where E is the p-adic field was treated in an earlier paper [same Proc. **55**, 1–9 (1952); these Rev. **13**, 728].
I. S. Cohen (Cambridge, Mass.).

Schwartz, Laurent. Sur les multiplicateurs de FL^p. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **22**, no. 21, 5 pp. (1953).

It is shown, using Riesz' theory of conjugate functions, that an operation that is essentially multiplication by the characteristic function of a projective polyhedron is continuous on FL^p to itself. Here FL^p is the collection of all distributions that are Fourier transforms of elements of L^p on euclidean space R^n, with the induced L^p norm, and a projective polyhedron is the intersection of R^n with a polyhedron in the projective space obtained from R^n by adjoining a hyperplane. If "projective polyhedron" is replaced by "unit sphere" the result fails for $n \geq 2$ and $p \geq 2n/(n-1)$ or $p < 2n/(n+1)$.
I. E. Segal (Chicago, Ill.).

Berman, D. L. Linear trigonometric polynomial operations in certain functional spaces. Doklady Akad. Nauk SSSR (N.S.) **88**, 9–12 (1953). (Russian)

In this note, the author generalizes slightly notions and theorems of an earlier note of his own [same Doklady (N.S.) **85**, 13–16 (1952); these Rev. **14**, 57] and of various notes by S. M. Lozinskiĭ [ibid. (N.S.) **64**, 453–456 (1949); these Rev. **10**, 529; and references cited in this review]. For a real-valued function f defined on $(-\infty, \infty)$, let $f_t(x) = f(x+t)$. Spaces E of real-valued functions on $(-\infty, \infty)$ satisfying the following axioms are considered. 1) The functions in E are periodic with period 2π and are summable on $[-\pi, \pi]$. 2) E is a normed linear space (not necessarily complete), with the usual definitions of sum and scalar multiplication. 3) If $f \in E$, then $f_t \in E$ for all real numbers t. 4) There exists a set $G = G_E$ of functions g, having period 2π and summable on $[-\pi, \pi]$ such that a) $f \in E$ and $g \in G$ imply that $\int_{-\pi}^{\pi} fg dx < \infty$; b) $\|f\|_E = \sup_{g \in G} \int_{-\pi}^{\pi} fg\, dx$, for all $f \in E$; c) if $g \in G$, then $g_t \in G$ for all real numbers t; d) if $g \in G$ and h is a measurable function of period 2π such that $|h(x)| \leq 1$ for almost all x, then $gh \in G$. 5) The set Q of all trigonometric polynomials is a dense subspace of E. Let E_1 and E_2 be two spaces satisfying axioms 1)–5). A mapping $U(f, x)$ carrying E_1 into E_2 is said to be a linear trigonometric polynomial operation of order n and type K if the following conditions hold. 1) $U(f, x)$ is a bounded linear operation. 2) For all $f \in E_1$, $U(f, x)$ is a trigonometric polynomial of order $\leq n$. 3) If $T \in E_1$ and T is a trigonometric polynomial of order $\leq n$, then

$$U(T, x) = \int_0^{2\pi} T(x+t) K(t) dt,$$

where $K(t)$ is a fixed trigonometric polynomial of order n. The principal theorem asserts that for every such operation $U(f, x)$, the equality

$$\frac{1}{2\pi} \int_0^{2\pi} U(f_t, x-t) dt = \int_0^{2\pi} f(x+t) K(t) dt$$

obtains. The proof is the same as the proof of the author's earlier, more special theorem [loc. cit.], which in turn is the same as a proof given by Marcinkiewicz [Acta Litt. Sci. Szeged **8**, 127–130 (1937)].
E. Hewitt.

Zaanen, A. C. Integral transformations and their resolvents in Orlicz and Lebesgue spaces. Compositio Math. **10**, 56–94 (1952).

The Orlicz spaces [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. Sci. Math. **1932**, 207–220] which include the L^p spaces on an interval Δ of euclidean n-space are based on the function $\Phi(u) = \int_0^u \varphi(u) du$ where $\varphi(u)$ is left continuous monotonic on $u \geq 0$, with $\varphi(0) = 0$. Associated with Φ is the function $\Psi(v) = \int_0^v \psi(v) dv$ where $\psi(v)$ is the left continuous inverse of φ, and we have $uv \leq \Phi(u) + \Psi(v)$. The additional condition (A): there exists an M such that, for $u \geq 0$, $\Phi(2u) \leq M\Phi(u)$, is frequently added. The L_Φ-space is defined as the set of functions f integrable on Δ, with finite Φ norm, where $\|f\|_\Phi = \sup \int_\Delta |fg|$ for all g for which $\int_\Delta \Psi |g| \leq 1$. The author proved previously [Ann. of Math. (2) **47**, 654–666 (1946); these Rev. **8**, 158] that an L_Φ space is linear complete. For functions of double variables $T(x, y)$ there are the spaces $L_{\Phi\Psi}$ of finite $\Phi\Psi$ norm, where $\|T\|_{\Phi\Psi} = \sup \int_\Delta \int_\Delta |T(x, y) f(y) g(x)| dy dx$ for all f such that $\int_\Delta \Phi |f| \leq 1$ and all g such that $\int_\Delta \Psi |g| \leq 1$. Such a space is also linear complete, and any such $T(x, y)$ defines through $\int_\Delta T(x, y) f(y) dy$ a bounded linear transformation on L_Φ to L_Φ. Finally there are the spaces D_Φ of functions $T(x, y)$ of finite Φ double norm $|||T|||_\Phi = \|t(x)\|_\Phi$ where

$$t(x) = \|T_x(y) - T(x, y)\|_\Psi.$$

(If $t(x)$ is not measurable, then $\|t(x)\|_\Phi = \inf \|t'(x)\|_\Phi$ for all $t'(x)$ measurable functions dominating $t(x)$.) The space D_Φ is linear normed complete and includes $L_{\Phi\Psi}$. If Φ satisfies condition (A), $T(x, y)$ is in D_Φ, and T is defined by $\int_\Delta T(x, y) f(y) dy$, then T^2 is completely continuous, generalizing a result of Dunford and Pettis [Proc. Nat. Acad. Sci. U. S. A. **25**, 544–550 (1939); Trans. Amer. Math. Soc. **47**, 323–392 (1940); these Rev. **1**, 57, 338] for L^p spaces. The

main portion of the paper is devoted to obtaining integral equation results in this more general setting. If $T(x, y)$ is in $L_{\Phi\Phi}$, then the transformation H_λ for λ in the resolvent set of $T-\lambda I$ is defined by $R_\lambda = -\lambda^{-1}I - \lambda^{-2}H_\lambda$, $R=(T-\lambda I)^{-1}$. Also, H_λ has an integral representation in terms of a function $H(x, y; \lambda)$ by the Neumann series, provided $|\lambda| > \|T_a\|$, T_a being the transformation defined by $|T(x, y)|$. If Φ satisfies condition (A), $T(x, y)$ is in $L_{\Phi\Phi}$, $T_x(y) = T(x, y)$ is in L_Ψ for almost all x of Δ, then such an integral representation exists for all $\lambda \neq 0$ in the resolvent set. If Φ satisfies (A), $T(x, y)$ is in $L_{\Phi\Phi}$, and $|||T^n|||_\Phi < \infty$ for some $n \geqq 1$, then for every $\lambda \neq 0$ in the point spectrum of T the reciprocal transformation H_λ has an integral representation with

$$H_\lambda(x, y) = \sum_{i=1}^{n-1} \lambda^{-i+1} T_i(x, y) + \lambda^{-n+1} K_\lambda(x, y)$$

with $|||K_\lambda|||_\Phi < \infty$, the $T_i(x, y)$ being the iterated kernels of T. By studying the character of the extension of the resolvent of a function $T(x, y)$ which belongs to D_Φ whose transpose belongs to D_Ψ in the vicinity of a point of the point spectrum, a generalized Fredholm determinant and first minor expressed in terms of the traces of the iterated kernels $T_i(x, y)$, $i \geqq 2$, are derivable, the resolvent kernel being as usual the quotient of these determinants. These are similar to those obtained by Smithies [Duke Math. J. **8**, 107–130 (1941); these Rev. **3**, 47] for the case where L_Φ is L^2 and the space D_Φ is the space of functions T for which $\int\int |T(x, y)|^2 dxdy < \infty$. *T. H. Hildebrandt.*

Plans Sanz de Bremond, Antonio. Some linear properties of bounded matrices. Revista Acad. Ci. Madrid **46**, 273–302 (1952). (Spanish. French summary)

A bounded (infinite) matrix A is called bicontinuous in a generalized sense if for every convergent sequence Y_n, belonging to the manifold of values Δ_A of A, there exists a convergent sequence X_n such that $AX_n = Y_n$ ($n=1, 2, \cdots$). Only real matrices are considered. Let $\mathfrak{A}$ be the class of bounded matrices, $\mathfrak{B}$ its subclass of generalized bicontinuous matrices, and $\mathfrak{C}$ the set of those members of $\mathfrak{A}$ not in $\mathfrak{B}$. Each of the classes $\mathfrak{B}$, $\mathfrak{C}$ is further divided into four subclasses, according as the rows and columns are (respectively) linearly independent or not, thus separating the matrices of $\mathfrak{A}$ into eight subclasses. These are discussed in the present work, and their relations to the Toeplitz classification [see e.g., Julia, Introduction mathématique aux théories quantiques, v. 2, Gauthier-Villars, Paris, 1938, especially pp. 206–215] are pointed out. Other results concern (i) the existence of right or left inverses for matrices of a given class; (ii) the determination of the class of the product of two or more matrices, given the class of each factor; (iii) the class of the minors of a matrix, and also of the matrix obtained by combining the rows (columns) of two or more matrices. *I. M. Sheffer* (State College, Pa.).

Bonsall, F. F., and Goldie, A. W. Algebras which represent their linear functionals. Proc. Cambridge Philos. Soc. **49**, 1–14 (1953).

Let A be an associative normed linear algebra over the reals, and K a subspace. Let t be a real function defined on the set of products ak such that $t(a_1k+a_2k) = t(a_1k)+t(a_2k)$ and $t(ak_1+ak_2) = t(ak_1)+t(ak_2)$. Then t is a "right trace over K." Let $kt(a) = t(ak)$. Then (t, K) is "effective" if the correspondence k with kt is one-to-one between K and the conjugate space A^* of A. (When A is finite-dimensional, the topological concepts are to be ignored.) Effective pairs (t, K) are constructed in several classes of algebras including the proper H^*-algebras of Ambrose. The concept is intended for the following extension theorem. Let H be a left-A module, and let H_0 be a submodule ("module" is here taken to imply also "linear space"). Let t be an effective right trace over some K in A. Let p be a positively homogeneous, subadditive real function on H such that $p(ax)$ (a fixed in A, x in H) is continuous in x. Let f be an A-homomorphism of H_0 to A, such that $t(f(x)) \leqq p(x)$ for x in H_0. Then f can be extended to an A-homomorphism F of H into A such that $t(aF(x)) \leqq p(ax)$. In the finite-dimensional case for A, it is shown that an effective right trace exists precisely when A is a Frobenius algebra. *R. Arens* (Los Angeles, Calif.).

Kondô, Motokiti. Les anneaux des opérateurs sur un espace de S. Banach et quelques problèmes qui s'y rattachent. I. J. Math. Tokyo **1**, 35–54 (1951).

An elementary consideration of the radical, semi-simplicity, direct sum decomposition, and representations of rings of operators on a Banach space. *W. Ambrose.*

Iseki, Kiyoshi. On B^*-algebras. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 12–14 (1953).

The author proves the well-known result that a B^*-algebra (1) is semi-simple, (2) is a C^*-algebra if and only if x^*x has a positive spectrum for each x, and (3) has no element x such that $x^*x+1=0$. *J. A. Schatz.*

Šreĭder, Yu. A. The structure of maximal ideals in rings of measures with convolution. Amer. Math. Soc. Translation no. **81**, 28 pp. (1953).

Translated from Mat. Sbornik N.S. **27**(69), 297–318 (1950); these Rev. **12**, 420.

Charles, Bernard. Sur certains anneaux commutatifs d'opérateurs linéaires. C. R. Acad. Sci. Paris **236**, 990 (1953).

Let E be a discrete vector space over a field K and R a commutative subring with unity of the ring S of all linear operators of E provided with a suitable convergence topology. Suppose that R is generated by the (complete) lattice $\varphi = \{P\}$ of the projections it contains. Let $\bar{R}$ be the closure of R in S, R' the commutator of R in S, R'' the center of R' (we have $R \subset \bar{R} \subset R'' \subset R'$). The following three theorems are stated. 1) E is the direct sum of subspaces E_α ($\alpha \in A$, A well ordered) each of which is reduced by every element of R. Every subspace of E reduced by R is of the form $P(E_\alpha)$. $P(E_\alpha)=0$ implies $P(E_\beta)=0$ for all $\beta > \alpha$. 2) $R'' = \bar{R}$. 3) $R' = \bar{R}$ if and only if every subspace of E reduced by R is of the form $P(E)$. *G. K. Kalisch.*

Schützenberger, Marcel Paul. Une interprétation de certaines solutions de l'équation fonctionnelle:

$$F(x+y) = F(x)F(y).$$

C. R. Acad. Sci. Paris **236**, 352–353 (1953).

In the ring of formal sequences of a non-commutative Banach algebra $\mathfrak{B}$, it is proposed to find a series

$$\mathrm{Exp}_u\,(x) = 1 + \sum_{n=1}^{\infty} x^n/a_1 \cdots a_n$$

for which identically:

$$\text{(1)} \qquad \mathrm{Exp}_u\,(ax)\,\mathrm{Exp}_u\,(by) = \mathrm{Exp}_u\,(ax+by)$$

for all a, b in the center of $\mathfrak{B}$ whenever x, y are weakly commutative; (2) $yx = uxy$ where u is an element of the

center of $\mathfrak{G}$ possessing an inverse. The required series is found in the form: (3) $\mathrm{Exp}_u(x)=\sum_{n=0}^{\infty}x^n/[n]_u!$ where $[n]_u!=\prod_{i=0}^{n}(1-u^i)/(1-u)$. It has the properties:

(4) $$\mathrm{Exp}_u(x)\,\mathrm{Exp}_{u^{-1}}(-x)=1;$$

(5) $$\lim_{\lambda\to 0}\,[\mathrm{Exp}_u(x+\lambda y)-\mathrm{Exp}_u(x)]/\lambda=y\,\mathrm{Exp}_u(x);$$

and (when the x_i are nilpotent and satisfy (2))

(6) $$\mathrm{Exp}_u(\sum x_i)=\prod(1+x_i).$$

A geometric interpretation is given. Let S be the incidence matrix in an h-dimensional projective geometry $\mathfrak{g}_h$ with coordinates in a $GF(p^n)$ and C the matrix corresponding to the relation of consecutive associate. Then $S=\mathrm{Exp}_{p^n}(C)$. Let $\mathfrak{g}'_{h'}$ be an h'-dimensional linear variety of $\mathfrak{g}_h$, S' the matrix corresponding to the incident relations which become equivalences under all homomorphisms of $\mathfrak{g}_h$ that annul $\mathfrak{g}'_{h'}$, C' the matrix of the consecutive associate relation, and $C''=C-C'$. Then

$$C''C'=p^nC'C'';$$
$$S=\mathrm{Exp}_{p^n}(C'+C'')=\mathrm{Exp}_{p^n}(C')\,\mathrm{Exp}_{p^n}(C'')=S'S''.$$

Particular cases are also discussed. *I. M. Sheffer.*

Saitô, Toshiya. Correction: On the measure-preserving flow on the torus. J. Math. Soc. Japan **4**, 338 (1952).

See same J. **3**, 279–284 (1951); these Rev. **14**, 59.

Salenius, T. Das Mass der kürzesten Linien in Kugelschalenräumen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **126**, 11 pp. (1952).

Let M be an n-dimensional ($n\geqq 3$) Riemannian which is topologically the product of a circle and an $(n-1)$-dimensional sphere. A geodesic g (geodesic ray r) on M is of class A if, on the universal covering manifold M^* of M, every finite segment of $g(r)$ yields an absolute minimum of arc-length on M^*. Let E denote the phase space of the geodesic flow on M, i.e., E is the $(2n-1)$-dimensional space of unit contravariant vectors on M. There is a natural measure μ defined in E such that the geodesic flow is measure preserving. It is proved that the set $E(A)$ consisting of those vectors in E which determine class A geodesic rays has μ-measure zero.

The space M^* can be considered as euclidean n-space with the origin deleted. A class A geodesic ray in M^* either approaches the origin or goes to infinity. The proof that $\mu E(A)=0$ follows readily from the theorem proved by the author in this paper that there cannot exist two class A geodesic rays with the same initial point, both of which go to the origin (or to infinity), and one of which is recurrent (stable in the sense of Poisson). *G. A. Hedlund.*

Calculus of Variations

Viola, Tullio. Sull'esistenza del minimo assoluto di taluni integrali multipli, connessi con i problemi al contorno per le funzioni iperarmoniche. Ann. Scuola Norm. Super. Pisa (3) **6**, 109–145 (1952).

For a bounded open region A in r-space with boundary $\mathfrak{F}A$ satisfying certain conditions of regularity the author exhibits a formula for the solution of $\Delta_2{}^nu=0$ which is such that prescribed values on $\mathfrak{F}A$ are assumed in a suitable limiting sense by u, $du/d\nu$, $\Delta_2{}^iu$, $d\Delta_2{}^iu/d\nu$ $(i=1,\cdots,p-1)$ in case $n=2p$, and by u, $du/d\nu$, $\Delta_2{}^iu$, $d\Delta_2{}^iu/d\nu$ $(i=1,\cdots,p-1)$, $\Delta_2{}^pu$ in case $n=2p+1$. The method of proof is an extension of that used by Fichera [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 34–39 (1951); these Rev. **13**, 758] for the case of $n=2$. *W. T. Reid* (Evanston, Ill.).

Chellevold, John O. Conjugate points of singular quadratic functionals for N dependent variables. Proc. Iowa Acad. Sci. **59**, 331–337 (1952).

The singular quadratic functionals have the form

$$(*)\qquad \liminf_{\epsilon\to 0}\int_{\epsilon}^{b}2\Omega(x,y,y')dx.$$

Here the n functions $y_i(x)$ are continuous on the closed interval $[0,b]$ and vanish at the ends, and are absolutely continuous on every closed subinterval of the half-open interval $(0,b]$. The integrand has the form

$$2\Omega=p_{ij}y_iy_j+2q_{ij}y_i'y_j+r_{ij}y_i'y_j',$$

where the coefficients p_{ij}, q_{ij}, r_{ij} are of class C' and the quadratic form $r_{ij}\pi_i\pi_j$ is positive definite on $(0,b]$. Conjugate points of 0 are defined following the paper by Morse and Leighton [Trans. Amer. Math. Soc. **40**, 252–286 (1936)], which treats the case $n=1$. The Jacobi condition for (*) is expressed as usual in terms of conjugate points. It is proved that when 0 is not its own conjugate point, all its conjugate points are given by the zeros of the determinant of a suitably chosen conjugate system of solutions of the Jacobi equations. It is also shown that 0 is its own conjugate point if and only if each point k on $(0,b]$ has an infinite number of conjugate points on $(0,k)$. The indicated proof for this last result seems to be insufficient, but is readily modified to remove this deficiency. *L. M. Graves.*

Sigalov, A. G. Two-dimensional problems of the calculus of variations. Amer. Math. Soc. Translation no. **83**, 121 pp. (1953).

Translated from Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 16–101 (1951); these Rev. **13**, 257, 1139.

Hammersley, J. M. On a conjecture of Nelder. Compositio Math. **10**, 241–244 (1952).

For the second order determinant $D=|\int_{0-}^{\infty}x^{i+j-2}e^{-x}dF(x)|$, $(i,j=1,2)$, where $F(x)$ is a distribution function of a non-negative variable [i.e., $F(x)$ is a non-decreasing function continuous on the right and satisfying $F(x)=0$, $x<0$; $F(x)\to 1$ as $x\to\infty$], the author states that J. A. Nelder has conjectured: (a) the maximum value of D is attained for the particular function $F(x)=F_0(x)$ defined as $F_0(x)=0$, $x<0$; $F_0(x)=\frac{1}{2}$, $0\leqq x<2$; $F_0(x)=1$, $x\geqq 2$, and (b) $F_0(x)$ is the unique function of the prescribed class for which D attains its maximum value. In the present paper the author proves conjecture (a), together with a weaker form of (b) to the effect that in the class of distribution functions of a non-negative variable which are step functions having commensurable saltuses the function $F_0(x)$ is the unique one for which D attains its maximum value.

W. T. Reid (Evanston, Ill.).

Koopman, Bernard O. The optimum distribution of effort. J. Operations Res. Soc. Amer. **1**, 52–63 (1953).

From the author's introduction: "When a limited amount of effort is available for the performance of two related tasks, the practical question of how it is to be divided between them in order to obtain the best over-all result is

one which constantly arises in operations research. The object of the present paper is to show how the question can often be put in quantitative form, and then to give it a simple mathematical solution in illustrative cases, interpreting the results in the language of recommended procedures. . . ."

The author restricts himself to the case where the effects are additive. *N. J. Fine* (Philadelphia, Pa.).

Theory of Probability

★Gillis, Paul P., von Mises, Richard, Ballieu, Robert, van Dantzig, D., Coutrez, Raymond, Bouckaert, L., Prigogine, I., Campus, F., Fauville, A., Fréchet, Maurice, Hirsch, Guy. Théorie des probabilités. Exposés sur ses fondements et ses applications. E. Nauwelaerts, Louvain; Gauthier-Villars, Paris, 1952. 195 pp. 200 Belgian francs.

A collection of papers, many expository, with the following titles: Gillis, On the notion of probability; v. Mises, On the foundations of the calculus of probability; Ballieu, Statistical analysis; v. Dantzig, Problems posed by the application of the calculus of probability; Coutrez, On the application of the calculus of probability to astronomy; Bouckaert, Statistical theories of physics; Prigogine, Probability and irreversibility; Campus, Probabilistic concept of the safety of edifices; Fauville, Problems posed by psychology to the calculus of probability; Fréchet, The calculus of probability in the social sciences; Hirsch, Conclusions. On the application of the theory of probability to statistics.
J. Wolfowitz (Ithaca, N. Y.).

Greenwood, Robert E. Probabilities of certain solitaire card games. J. Amer. Statist. Assoc. **48**, 88–93 (1953).

The probabilities of coincidences in an ordinary deck of 52 cards, considered apart from suit and hence of specification (4^{13}), are evaluated in a straightforward manner. The results are obtained more easily by the asymptotic formula given by Cattaneo [Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **101**, 89–104 (1942); these Rev. **8**, 247] and modified by the reviewer in the cited review. Also, results of 1000 trials are compared with those calculated. It may be noted that the author's procedure in evaluating the probabilities is unusual but not new; it appears in MacMahon [Combinatory analysis, v. 1, Cambridge, 1915, Section III, Chapter III]. *J. Riordan.*

Waugh, Dan F., and Waugh, Frederick V. On probabilities in bridge. J. Amer. Statist. Assoc. **48**, 79–87 (1953).

Bayes formula is used to calculate certain bridge probabilities as they are affected by actual play, two examples being given. It is argued that such calculations are more significant than the usual ones which do not include information obtained during play. In the examples, the assignment of prior (unconditional) probabilities of card holdings is taken as unambiguous. *J. Riordan* (New York, N. Y.).

Ladegast, Konrad. Einige Abschätzungen für endliche unstetige Verteilungen. Mitteilungsblatt Math. Statist. **5**, 75–86 (1953).

Let $\nu_r(a)$ denote the rth absolute moment about the point a of a discrete chance variable taking a finite number of values. The author derives several inequalities of the Čebyšev type involving $\nu_r(a)$. The following is a typical example: Let $0<b_1<b_2<\max|x-a|$; then

$$(b_2^r-b_1^r)\Pr[|x-a|>b_2]-b_1^r\Pr[|x-a|\leq b_1]<\nu_r(a)-b_1^r.$$

Also, relations among the mean, median, and mode of the distribution are obtained. *G. E. Noether.*

af Hällström, Gunnar. Ein lineares Inselproblem der kombinatorischen Wahrscheinlichkeitsrechnung. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **123**, 9 pp. (1952).

Probability distribution and expected value for the familiar problem of number of runs among m objects thrown at random onto n points of a line or circle. Some computations of these properties are also given for the maximum number of runs encountered at any moment along the way if n balls are thrown one at a time. *J. Kiefer.*

Aczél, J. On composed Poisson distributions. III. Acta Math. Acad. Sci. Hungar. **3**, 219–224 (1952). (Russian summary)

This is a continuation of some recent work by Jánossy, Rényi and the author [same Acta **1**, 209–224 (1950); **2**, 83–98 (1951); these Rev. **13**, 663]. Let $w_k(t_1, t_2)$ be the probability that exactly k events occur during the time interval (t_1, t_2). It is assumed that the numbers of events occurring in nonoverlapping intervals are independent. The form of $w_k(t_1, t_2)$ is derived inductively from a Chapman-Smoluchowsky type equation. *E. Lukacs.*

Metropolis, N., and Ulam, S. A property of randomness of an arithmetical function. Amer. Math. Monthly **60**, 252–253 (1953).

Given a set E, and a transformation $y=f(x)$ on E to E, let λ_i represent the number of points y such that for each y there are exactly i distinct points x satisfying $y=f(x)$. If the set contains $n<\infty$ elements and n is large, and if the association defined by f is random, then the λ_i should have a Poisson distribution. The authors have examined a particular (nonrandom) association used in Monte Carlo calculations to produce pseudo-random digits, and found the distribution very nearly Poisson. The association is defined as follows: x is an integer expressible in binary form with k digits; $f(x)$ is the integer y expressed by the middle k digits of x^2. *A. S. Householder* (Oak Ridge, Tenn.).

Duparc, H. J. A., Lekkerkerker, C. G., and Peremans, W. Reduced sequences of integers and pseudo-random numbers. Math. Centrum Amsterdam. Rapport **ZW 1953-002**, 15 pp. (1953).

The authors discuss sequences of integers whose least positive residues modulo m are to be used as a source of pseudo-random digits. Two kinds of such sequences are discussed: the general geometric progression and the Fibonacci sequence. The authors investigate only the length of the period of a sequence and do not touch upon the ability of the corresponding pseudo-random digits to pass any of the standard tests. The period length of a geometric progression is the exponent, modulo m, of the common ratio of the progression while that of the Fibonacci sequence is the product of Lucas' "rank of apparition of m" by 1, 2, or 4. There are 6 illustrative tables. For some applications care must be used to eliminate certain contagious influences in the Fibonacci case as well as in the case of the geometric progression of small ratio. *D. H. Lehmer.*

Blomqvist, N. On an exhaustion process. Skand. Aktuarietidskr. 35 (1952), 201–210 (1953).

There are n mines on a path; m persons go through it, one after another. Each has a constant probability $p=1-q$ of hitting a previously unhit mine and his chance is independent of his predecessors'. Let the total number of hits be $X(m, n)$. It is shown that

$$P(X=x)=q^{(m-x)(n-x)}F(m-x+1, m)F(n-x+1)/F(1, x) \quad \text{for } 0\leqq x\leqq \min(m, n),$$

where $F(a, b)=\prod_{i=a}^{b}(1-q^i)$. From this various limiting forms are obtained. Different ways of passing to a continuous parameter process is discussed. If M is the number of persons required to hit all the mines, the limiting distribution and mathematical expectation of M under various suitable normings are given explicitly. The lemma is known [see Hardy and Wright, Introduction to the theory of numbers, Oxford, 1938, p. 278]. *K. L. Chung.*

Chung, K. L. Sur les lois de probabilité unimodales. C. R. Acad. Sci. Paris 236, 583–584 (1953).

Un théorème de A. I. Lapin affirme que la convolution de deux lois de probabilités unimodales avec sommet 0 est unimodale de sommet 0: l'auteur en montre l'inexactitude par des contre exemples. Par suite le théorème de B. Gnedenko selon lequel toute loi de la classe L est unimodale et dont la démonstration repose sur le théorème de Lapin reste à prouver. *R. Fortet* (Paris).

Lenz, Hanfried. Über die Cramérschen asymptotischen Entwicklungen der Wahrscheinlichkeitsrechnung. Math. Ann. 125 (1952), 307–313 (1953).

The asymptotic expansion of the characteristic function used in work of Cramér and P. L. Hsu [Ann. Math. Statistics 16, 1–29 (1945); these Rev. 6, 233] is obtained by the addition theorem for Hermite polynomials.
K. L. Chung (Ithaca, N. Y.).

Krishna Iyer, P. V. Factorial moments and cumulants of distributions arising in Markoff chains. J. Indian Soc. Agric. Statistics 4, 113–123 (1952).

Formulas for computing the items in the title for 0-or-1 random variables with applications to distributions of numbers of various types of configurations which arise in throwing black or white balls at random onto points of a line or plane lattice (e.g., number of black-black joins, etc.).
J. Kiefer (Ithaca, N. Y.).

Rényi, A. On projections of probability distributions. Acta Math. Sci. Hungar. 3, 131–142 (1952). (Russian summary)

The following theorem is due to Radon: Let D be a bounded domain, and $f(x, y)$ a continuous function defined in it. Assume that the integral of $f(x, y)$ vanishes along every chord of $f(x, y)$; then $f(x, y)$ is identically 0. The author first of all points out that Radon's theorem follows from the following result due to Cramér and Wold: Let $F(x, y)$ be a distribution function. Then $F(x, y)$ is uniquely determined by the values of its projections on all the straight lines through the origin, i.e., by the values of the integrals.

$$F_{l_\varphi}(p)=\iint_{x\cos\varphi+y\sin\varphi\leq p} dF(x, y),$$

where φ denotes the angle between the line l_φ and the x-axis.

The author then gives Hajós's proof for his conjecture: A discrete mass distribution consisting of n distinct mass points is completely determined if its projections on $n+1$ arbitrary different straight lines through the origin are given. The author points out that this theorem is best possible, i.e., $n+1$ cannot be replaced by n; further, he discusses three-dimensional generalizations.

The author further raises the following problem: Put $f(x, y)=1/\pi(1-x^2-y^2)^{1/2}$ for $x^2+y^2<1$ and $f(x, y)=0$ for $x^2+y^2\geqq 1$. Then clearly the integral of $f(x, y)$ on every chord of the unit circle is 1. Is there any other curve of constant breadth for which there exists a function $f(x, y)$ whose integral is constant on every chord? *P. Erdös.*

Diananda, P. H. Some probability limit theorems with statistical applications. Proc. Cambridge Philos. Soc. 49, 239–246 (1953).

A sequence $X_1, X_2, \cdots$ of random variables is called m-dependent if the sets $(X_1, \cdots, X_r)$ and $(X_s, \cdots, X_n)$ are independent whenever $s-r>m$. Let $\{X_i\}$ be m-dependent, stationary, with zero means and finite variances. Then $n^{-\frac{1}{2}}(X_1+\cdots+X_n)$ has a normal limiting distribution. This sharpens a result of Hoeffding and Robbins [Duke Math. J. 15, 773–780 (1948); these Rev. 10, 200]. Let $A_0, A_1, \cdots$ be scalar constants such that $\sum|A_i|$ converges and let $U_i=\sum_{t=0}^{\infty}A_tX_{i-t}$. Then $n^{-\frac{1}{2}}(U_1+\cdots+U_n)$ has a normal limiting distribution. Extensions to vector variables are considered. The results are applied to automoments, serial correlation coefficients and certain extensions of Quenouille's test for autoregressive schemes [see Bartlett and Diananda, J. Roy. Statist. Soc. Ser. B. 12, 108–115 (1950); these Rev. 12, 512]. *W. Hoeffding* (Chapel Hill, N. C.).

Chung, K. L. Corrections to my paper "Fluctuations of sums of independent random variables". Ann. of Math. (2) 57, 604–605 (1953).

See same Ann. (2) 51, 697–706 (1950); these Rev. 11, 731.

Zadeh, L. A. On a class of stochastic operators. J. Math. Physics 32, 48–53 (1953).

For each λ, let $\{H(t, \lambda), -\infty<t<\infty\}$ be a stationary (wide sense) stochastic process. The author considers the stochastic process defined by the integral $\int_{-\infty}^{\infty}H(t,\lambda)e^{it\lambda}dF(\lambda)$. If $\{F(\lambda), -\infty<\lambda<\infty\}$ is itself a stochastic process, with orthogonal increments, independent of the $H(t, \lambda)$ family for all t, λ, the integral defines a process stationary in the wide sense whose covariance function is evaluated. This evaluation, which has a simple operational interpretation, was announced in an earlier paper [same J. 30, 73–78 (1951); these Rev. 13, 458]. The discussion is carried through purely formally. *J. L. Doob* (Urbana, Ill.).

Bochner, S. Stochastic processes with finite and non-finite variance. Proc. Nat. Acad. Sci. U. S. A. 39, 190–197 (1953).

Let $\{x(t), t\in T\}$ be a stochastic process (family of random variables). Then it is well known that, if the $x(t)$'s have finite second moments, there is a corresponding stochastic process with the same parameter set T whose random variables have (jointly) Gaussian distributions with the same first and second moments. The author extends this theorem to the corresponding one for his generalized stochastic processes [same Proc. 36, 439–443 (1950); these Rev. 12, 425], in which he supposes Euclidean component spaces, with affine connections. If the second moments of the distributions in the generalized processes do not exist, but if the

distributions involved are infinitely divisible, the Gaussian and non-Gaussian components can be separated to yield two new generalized processes.

If $0<q<2$, and if ϕ is the characteristic function of a distribution, it is proved that the absolute qth moment of the distribution is finite if and only if

$$\lim_{\epsilon\to 0}\int_\epsilon^1\left[1-\frac{\varphi(\alpha)+\varphi(-\alpha)}{2}\right]\frac{d\alpha}{\alpha^{q+1}}$$

exists and is finite. This is given as an application of a simple identity connecting distribution functions and their characteristic functions. A similar application is made in the multivariate case. *J. L. Doob* (Urbana, Ill.).

Karhunen, Kari. Zur Interpolation von stationären zufälligen Funktionen. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **142**, 8 pp. (1952).

Let $\{x(t), -\infty<t<\infty\}$ be a stationary stochastic process which has no deterministic component. If $a<b$, let L_2 [$L_2(a, b)$] be the closed linear manifold of random variables spanned by the $x(t)$'s [for $t\leqq a$, $t\geqq b$]. If every $x(t)$ lies in $L_2(a, b)$, that is, if $L_2=L_2(a, b)$, the process is interpolatable over (a, b). The author finds a simple representation for the elements of the orthogonal complement of $L_2(a, b)$ relative to L_2, and gives illustrative examples. *J. L. Doob.*

Kinney, J. R. Continuity properties of sample functions of Markov processes. Trans. Amer. Math. Soc. **74**, 280–302 (1953).

Se donnant la probabilité de passage

$$P(t, x; \tau, e)=\Pr[x(\tau)\in e\,|\,x(t)=x](t<\tau)$$

d'un processus de Markoff permanent $X(t)$, l'auteur obtient un jeu de conditions sur $P(t, x; \tau, e)$, du même type mais non équivalentes, et suffisantes pour: (a) que $X(t)$ soit presque-sûrement une fonction continue, à un nombre fini de sauts près; (b) que $X(t)$ soit presque-sûrement une fonction continue; par exemple, si pour

$$h\to+0,\quad \Pr[|X(t+h)-X(t)|>\varepsilon\,|\,X(t)=x]$$

est un infiniment petit d'ordre plus grand que 1, et cela avec une certaine uniformité en (t, x), $X(t)$ est presque-sûrement une fonction continue. Il n'est pas possible d'indiquer ici avec précision les résultats de l'auteur, qui reposent sur des théorèmes de J. L. Doob sur les martingales, et qui dépassent nettement ceux du même genre obtenus jusqu'à présent. *R. Fortet* (Paris).

Matschinski, Matthias. Equations générales des processus stochastiques. Population "holostochastique" et population "semistochastique". C. R. Acad. Sci. Paris **236**, 580–583 (1953).

Kunisawa, Kiyonori. The mathematical foundation of Shannon's information source and its application to binary coding. (A statistical treatment of binary coding.) Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **2**, no. 1, 4–26 (1952).

The author applies asymptotic Markov chain theorems to sequences of symbols generated by multiple Markov chains, and discusses the implications for the coding of long sequences. *J. L. Doob* (Urbana, Ill.).

Grosjean, Carl C. Solution of the non-isotropic random flight problem in the k-dimensional space. Physica **19**, 29–45 (1953).

This paper is devoted to a generalization of the classical problem of random flights to k-dimensional space when the probability distribution of the length (l) and direction (γ) of the $(n+1)$th flight is further not spherical with respect to the nth flight. The author's method can be illustrated by considering the two-dimensional problem. In this latter case we consider the probability distribution of the sum of N vectors in a plane when the probability distributions of their lengths are given by $f_n(l)$ $(l>0; n=1, 2, 3, \cdots)$ and the probability distribution of the angle γ between the nth and the $(n+1)$th vectors is given by $p_n(\gamma)/2\pi$. Let $\varphi_n(r, \theta)drd\theta/2\pi$ denote the elementary probability that the end point of the nth vector lies between r and $r+dr$ and makes an angle θ with the polar axis and $\psi_n(r, \theta)drd\theta/2\pi$ the elementary probability that the end point of the nth vector lies between r and $r+dr$ and the $(n+1)$th vector makes an angle between θ and $\theta+d\theta$ with the polar axis; the author then shows that φ_n and ψ_n satisfy recurrence relations

$$(1)\qquad \psi_n(r, \theta)=\int_0^{2\pi}\varphi_n(r, \alpha)p_n(\gamma)d\alpha/2\pi,$$

where $\gamma=\theta-\alpha$ for $0\leqq\alpha\leqq\theta$ and $2\pi+\theta-\alpha$ for $\theta\leqq\alpha\leqq 2\pi$, and

$$(2)\qquad \varphi_n(r, \alpha)=r\int_0^\infty \psi_{n-1}(x, \theta)f_n(y)dy/x$$

where $r^2=x^2+y^2+2xy\cos\theta$. Expanding $\varphi_n(r, \theta)$, $\psi_n(r, \theta)$ and $p_n(\gamma)$ in Fourier series in the forms

$$(3)\qquad \varphi_n(r, \theta)=\sum\xi_n^{(h)}(r)e^{ih\theta},\quad \psi_n(r, \theta)=\sum\chi_n^{(h)}e^{ih\theta},$$
$$p_n(\gamma)=\sum A_n^{(h)}e^{ih\gamma}$$

and further representing $\xi_n^{(h)}(r)$ by the integral

$$(4)\qquad \xi_n^{(h)}(r)=r\int_0^\infty w_n^{(h)}(u)J_h(rw)dw$$

(where J denotes the Bessel function) the author shows that the relations (1) and (2) become

$$(5)\qquad \chi_n^{(h)}(r)=A_n^{(h)}\xi_n^{(h)}(r)\quad (n=1, 2, 3, \cdots; h=0, \pm1, \cdots),$$

and

$$(6)\qquad w_n^{(h)}(u)=u\sum_m F^{(h-m)}(u)\int_0^\infty \chi_{n-1}^{(m)}(x)J_m(xu)dx$$
$$=\sum_m A_{n-1}^{(m)}w_{n-1}^{(m)}(u)F_n^{(h-m)}(u)$$
$$(n=2, 3, \cdots; h=0, \pm1, \cdots)$$

where $F_n^{(h)}(u)=\int_0^\infty f_n(y)J_h(yu)dy$. Equations (5) and (6) lead directly to a formal solution of the problem. The author considers various special cases of the resulting solution and further generalizes it for k-dimensional space.
S. Chandrasekhar (Williams Bay, Wis.).

Pollaczek, Félix. Sur une généralisation de la théorie des attentes. C. R. Acad. Sci. Paris **236**, 578–580 (1953).

The author's extensive studies of telephone traffic in both non-delay and delay systems [Ann. Inst. H. Poincaré **11**, 135–173 (1949); these Rev. **11**, 672 and the further references given there] have started from the assumption that n arrivals appear at random in time interval T. Here this is replaced by the more general assumption that the intervals between arrivals are independent random variables with a common distribution function. Taking the holding (service) times also as independent random variables with a common distribution function, and supposing service in order of arrival, the determination of the interesting variables of the problem in a delay system with s servers, the delay of the nth arrival and the congestion at its moment of arrival, is reduced to the solution of s simultaneous integral equa-

tions, of the same form for all variables and in which the system specification appears through the Laplace-Stieltjes transforms of the two distribution functions. More precisely, the variables directly determined are the generating functions of the delay and congestion mentioned above. For exponential distribution of service times, the solutions of the system of integral equations for these last variables are given explicitly; and it is noted that for any service time distribution whose transform is rational, the solution may be obtained in a finite number of operations.

It is said that the limiting delay and congestion, (for n increasing) which correspond to the stationary state, are qualitatively equivalent to those obtained independently by D. G. Kendall (to be published). It may also be noticed that for a single server results have previously been given by D. V. Lindley [Proc. Cambridge Philos. Soc. **48**, 277–289 (1952); these Rev. **13**, 759]. Finally it is said that the further assumption introduced in another paper [C. R. Acad. Sci. Paris **234**, 1246–1248 (1952); these Rev. **13**, 761] that calls obliged to wait incur a further delay of arbitrary distribution may be accommodated by relatively slight changes.

J. Riordan (New York, N. Y.).

Pollaczek, Félix. Généralisation de la théorie probabiliste des systèmes téléphoniques sans dispositif d'attente. C. R. Acad. Sci. Paris **236**, 1469–1470 (1953).

The generalization consists of allowing intervals between arrivals, as well as service times, to be independent random variables with a common distribution function. A previous note [see the preceding review] has given results for delay systems; here the results are for non-delay systems (in the telephone trunking application, calls arriving when all trunks are busy are dismissed at once and "lost" to the system). As before, the problem is reduced to the solution of s simultaneous equations; s is the number of trunks (servers). By choice of the input (left-hand side) of these equations, various congestion variables are determined, like p_n^a, the probability that the nth call finds a trunks occupied, $p_{n,j}^{a,b}$, the probability call n finds a and call $n+j$ finds b trunks occupied, and so on. More directly, the generating functions for these probabilities are found. Explicit results for the simplest of these generating functions are given for exponential service time distribution, and are seen to be concordant with known results in limit cases.

There seem to be a few typographical errors; in the second form of equation (4) $\epsilon_1(-k)$ should be $\epsilon_2(-k)$, and in the last equation the factor $(p-z)$ in the denominator should probably be $(p-z)^2$. Further, a product which appears probably needs the usual convention of being unity when no factors are indicated.

J. Riordan.

Messel, H. The solution of the fluctuation problem in nucleon cascade theory: homogeneous nuclear matter. Proc. Phys. Soc. Sect. A. **65**, 465–472 (1952).

The novel idea which this paper introduces in the theory of nucleon cascades is to write a diffusion equation involving the concept of the "last collision". Thus if $C_n(\epsilon_1, \cdots, \epsilon_n; x)d\epsilon_1 \cdots d\epsilon_n$ is the differential probability of finding one particle in each of the energy intervals $\epsilon_i, \epsilon_i + d\epsilon_i$ and any number of particles with arbitrary energies at a depth x in homogeneous nuclear matter due to a primary nucleon of unity energy, then C_n must satisfy a recurrence relation of the form

$$(1)\quad C_n(\epsilon_1, \cdots, \epsilon_n; x) = \int_0^x d\theta \Big\{ \int_0^\infty \sum_{l=1}^n C_n(\epsilon_1', \cdots, \epsilon'_{n-1}, \epsilon; x-\theta) W\Big(\frac{\epsilon_l}{\epsilon}\Big)\frac{d\epsilon}{\epsilon} + \int_0^\infty \sum_{j\neq l} C_{n-1}(\epsilon_1', \cdots, \epsilon'_{n-2}, \epsilon; x-\theta) \times w\Big(\frac{\epsilon_j}{\epsilon}, \frac{\epsilon_l}{\epsilon}\Big)\frac{d\epsilon}{\epsilon^2}\Big\} e^{-n\theta},$$

where the primes attached to the ϵ's signify that among $(\epsilon_1', \cdots, \epsilon'_{n-1})$, ϵ_l is not included and similarly among $(\epsilon_1', \cdots, \epsilon'_{n-2})$ neither ϵ_j nor ϵ_l is included. Further in (1) $w(\epsilon_j/\epsilon, \epsilon_l/\epsilon)$ (symmetrical in j and l) is related to the probability that a nucleon of energy ϵ produces on collision secondary and recoil nucleons of energies ϵ_j and ϵ_l; and finally

$$W(\epsilon) = 2\int_0^\infty w(\epsilon_1, \epsilon)d\epsilon_1.$$

Equation (1) expresses the fact that n particles with energies in the interval $\epsilon_i, \epsilon_i + d\epsilon_i$ $(i=1, \cdots, n)$ at depth x can have resulted in only one of the two ways, namely, (i) at depth $x-\theta$ $(0 \leqq \theta \leqq x)$, $n-1$ of the particles may have attained their final energies and one of the particles undergoes a collision giving rise to the nth particle and after this no further change takes place, and (ii) at depth $x-\theta$, $(n-2)$ particles have their final energies and one of the particles undergoes a collision and gives rise to the $(n-1)$th and the nth particles. The two terms on the right-hand side of (1) correspond to these two alternatives. Differentiating (1) one obtains the diffusion equation

$$(2)\quad \frac{\partial}{\partial x} C_n(\epsilon_1, \cdots, \epsilon_n; x) = -nC_n(\epsilon_1, \cdots, \epsilon_n; x) + \int_0^\infty \sum_{l=1}^n C_n(\epsilon_1', \cdots, \epsilon'_{n-1}, \epsilon; x) W\Big(\frac{\epsilon_l}{\epsilon}\Big)\frac{d\epsilon}{\epsilon} + \int_0^\infty \sum_{j\neq l} C_{n-1}(\epsilon_1', \cdots, \epsilon'_{n-2}, \epsilon; x) w\Big(\frac{\epsilon_j}{\epsilon}, \frac{\epsilon_l}{\epsilon}\Big)\frac{d\epsilon}{\epsilon^2}.$$

Applying to (2) an n-fold Mellin transformation one obtains

$$(3)\quad \frac{\partial}{\partial x} V_n(s_1, \cdots, s_n; x) = -nV_n(s_1, \cdots, s_n; x) + \sum_{l=1}^n W^*(s_l) V_n(s_1, \cdots, s_n; x) + 2\sum_{j\neq l} W(s_j, s_l) V_{n-1}(s_1', \cdots, s'_{n-2}, s_j + s_l; x),$$

where $V_n(s_1, \cdots, s_n; x)$, $W(s_j, s_j)$, and $W^*(s)$ are the n-fold, two-fold, and simple Mellin transforms of $C_n(\epsilon_1, \cdots, \epsilon_n; x)$, $w(\epsilon_j, \epsilon_l)$, and $W(\epsilon)$, respectively. The author next shows that the solution of (3) can be written in the form

$$(4)\quad V_n(s_1, \cdots, s_n; x) = 2\exp\Big\{-x\sum_{i=1}^n \alpha(s_i)\Big\} \int_0^x d\theta \exp\Big\{\theta \sum_{i=1}^n \alpha(s_i)\Big\} \times \sum_{j\neq l} W(s_j, s_l) V_{n-1}(s_1', \cdots, s'_{n-2}, s_j + s_l; \theta),$$

where $\alpha(s) = 1 - W^*(s)$. The rest of the paper is devoted to showing how all the results derived earlier form a direct solution of the usual cascade equations [cf. H. Messel, Communications Dublin Inst. Advanced Studies. Ser. A. no. 7 (1951); these Rev. **14**, 563] can be obtained quite simply from the foregoing solution (4).

S. Chandrasekhar (Williams Bay, Wis.).

Messel, H., and Potts, R. B. The solution of the fluctuation problem in a finite absorber for nucleon cascades. Proc. Phys. Soc. Sect. A. 65, 473–480 (1952).

The concept of "last collision" introduced in the paper reviewed above is further exploited to solve the problem of nucleon cascade in a finite absorber. Thus, letting $H_n(E_0; E_1, \cdots, E_n; \theta)dE_1\cdots dE_n$ denote the differential probability that after a depth θ (measured in suitable units) a primary nucleon of energy E_0 has given rise to n nucleons with energies in the range E_j, E_j+dE_j $(j=1, \cdots, n)$ in any order and letting $Q_n(E_0; E_1, \cdots, E_n)dE_1\cdots dE_n d\theta$ denote the probability that a nucleon of energy E_0 collides with a nucleon in traversing a distance $d\theta$ and gives rise to nucleons with energies in the range E_j, E_j+dE_j $(j=1, \cdots, n)$ in any order, the author shows by considering the last collision that the diffusion equation satisfied by H_n is

$$(1)\quad \frac{\partial}{\partial\theta}H_n+nH_n=\sum_{k=1}^{n}\sum_{(k)}\int_0^\infty Q_k(\eta; E_1', \cdots, E_k') \times H_{n-k+1}(E_0; E'_{k+1}, \cdots, E_n', \eta; \theta)d\eta,$$

where $\sum_{(k)}$ signifies summation over all decompositions of $E_1, \cdots, E_n$ into two groups $E_1', \cdots, E_k'$; $E'_{k+1}, \cdots, E_n'$. By taking the n-fold Mellin transformation with respect to energies and a single Laplace transformation with respect to θ, we obtain from (1) the equation

$$(2)\quad (p+n)R_n(s_1, \cdots, s_n; p)=\sum_{k=1}^{n}\sum_{(k)}N_k(s_1', \cdots, s_k') \times R_{n-k+1}(s'_{k+1}, \cdots, s_n', s_1'+\cdots+s_k'; p)$$

for the corresponding transforms R_n and N_k of H_n and Q_k, respectively. The solution of (2) is expressed in the form

$$(3)\quad (p+h)R_{n+1} =\sum_C \prod_{j=m}^{1} \sum_{\text{Comb}} N_{n(j)+1}(s_{q(j-1)+1}, \cdots, s_{q(j)}, s_{q(j)+1}+\cdots+s_{n+1}) \times\{p+[q(j)+1]h\}^{-1},$$
$$(p+h)R_1=1,$$

and $h=1-2\{1-(1+D_A)\exp(-D_A)\}/D_A^2$, where D_A denotes the diameter of the absorbing nucleus of atomic number A. In (3) $\sum_C$ signifies the sum over the 2^{n-1} decompositions of n; C is the decomposition $n(1), n(2), \cdots, n(m)$, $\sum_{j=1}^m n(j)=n$, m is the number of nucleon-nucleus collisions, $n(j)+1$ gives the number of nucleons arising from the jth nucleon-nucleus collision; $q(j)=n(1)+\cdots+n(j)$, $q(0)=0$, and $q(m)=n$; and finally $\sum_{\text{Comb}}$ signifies summation over all combinations $C_{n(j)+1}^{q(j)+1}$ of the $q(j)+1$ symbols $s_1, s_2, \cdots, s_{q(j)}$, $s_{q(j)+1}+\cdots+s_{n+1}$ taken $n(j)+1$ at a time.

The required solution is then obtained by taking the inverse Laplace-Mellin transform of (3). The solution given in (3) is illustrated by considering the case $R_4(s_1, s_2, s_3, s_4; p)$.

Methods similar to the foregoing are also applied to the solution of the equation (again derived from a consideration of the "last collision")

$$(4)\quad \frac{\partial}{\partial\theta}J_n+nJ_n=\sum_{k=1}^{n}\sum_{r=k}^{\infty}\sum_{(k)}\frac{1}{(r-k)!}\int_0^\infty d\eta\int_0^\infty d\xi_1\cdots\int_0^\infty d\xi_{r-k} \times Q_r(\eta; E_1', \cdots, E_k', \xi_1, \cdots, \xi_{r-k}) \times J_{n-k+1}(E_0; E'_{k+1}, \cdots, E_n', \eta; \theta),$$

where $J_n(E_0; E_1, \cdots, E_n; \theta)dE_1\cdots dE_n$ denotes the probability of finding n particles with energies E_j, E_j+dE_j at depth θ and any number of particles with arbitrary energies due to a primary nucleon of energy E_0; also $\sum_{(k)}$ signifies the summation over all decompositions of $E_1, \cdots, E_n$ into two groups $E_1', \cdots, E'_{n-k}$ and $E'_{n-k+1}, \cdots, E_n'$.

S. Chandrasekhar (Williams Bay, Wis.).

Messel, H., and Potts, R. B. Note on the fluctuation problem in cascade theory. Proc. Phys. Soc. Sect. A. 65, 854–856 (1952).

In the theory of nucleon cascades one introduces the functions $C_n(E_0; E_1, \cdots, E_n; \theta)$, $H_n(E_0; E_1, \cdots, E_n; \theta)$ and $J_n(E_0; E_1, \cdots, E_n; \theta)$ (for the definitions of these functions see the two preceding reviews). In addition to these functions, one also introduces in the theory the function $\phi(E_0; E, N, \theta)$ which gives the probability of finding N particles with energies greater than E and an arbitrary number of particles with energies less than E at a depth θ. If $T_n(E_0; E, \theta)$ denotes the moment

$$T_n(E_0; E, \theta)=\sum_{a=0}^{\infty}\frac{(n+a)!}{a!}\phi(E_0; E, n+a, \theta),$$

then the authors show by making use of the equations governing H_n and J_n (equations (1) and (4) of the preceding review) that

$$T_n(E_0; E, \theta)=\int_E^\infty dE_1\cdots\int_E^\infty dE_n J_n(E_0; E_1, \cdots, E_n; \theta).$$

With the solution of J_n given in the earlier paper [see the preceding review] the foregoing relation suffices to determine T_n. *S. Chandrasekhar* (Williams Bay, Wis.).

Lefèvre, J. Application de la théorie collective du risque à la réassurance "Excess-Loss". Skand. Aktuarietidskr. 35 (1952), 160–187 (1953).

If P denotes the total net risk-premiums received up to a given point of time and χP $(\chi>1)$ the maximum amount of claims an insurance company is prepared to sustain, the "excess-loss" premium is given by $\int_{\chi P}^\infty(x-\chi P)dF(x, P)$, where $F(x, P)$ is the distribution function of claims x at "time" P. The author applies asymptotic approximations of the Edgeworth type [Cramér, Random variables and probability distributions, Cambridge, 1937] to obtain two sets of numerical premium values, one based on an exponential distribution of sums at risk and the other on an actual observational distribution. The two-term approximations are the most useful. *H. L. Seal.*

Sverdrup, Erling. Basic concepts in life assurance mathematics. Skand. Aktuarietidskr. 35 (1952), 115–131 (1953).

The author proposes a probabilistic basis for actuarial mathematics. His fundamental random variable is the age at death X, the distribution of X is denoted by $G(x)$; this is the probability that a person of age 0 will die not later than at age x. The author considers a group of N persons and makes the following assumptions: (A) The random variables $X_1, X_2, \cdots, X_N$ are independently and identically distributed with common distribution $G(x)$; (B) there exists a finite ω such that $G(\omega)=1$. On the basis of these assumptions the probabilities used in actuarial mathematics are introduced and two theorems are proved which justify the principle of equivalence. Finally, suggestions are made concerning the application of statistical methods to the estimation of probabilities of death and to the graduation of mortality tables. *E. Lukacs* (Washington, D. C.).

Mathematical Statistics

Taguti, Gen-iti. Tables of 5% and 1% points for the Polya-Eggenberger's distribution function. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 2, no. 1, 27 (5 plates) (1952).

The Polya-Eggenberger distribution function is defined by

$$F(k;h,d)=\sum_{n=0}^{k-1}\frac{\Gamma(h/d+n)}{\Gamma(h/d)}\frac{d^n}{n!}(1+d)^{-h/d-n},\quad k\geq 1,\ h>0,\ d>0.$$

(The author's expression has the misprint k in place of $k-1$ in the above summation.) Table 1 gives the minimum values of h to 3 significant figures such that $F(k;h,d)\geq .95$ and $F(k;h,d)\geq .99$ for $h/d=.5(.5)15$, 20, 30, 60, ∞, and $k=1(1)25$. Table 2 gives the values of $1/h$ to 3 significant figures for $d/h=0$, .05, .1(.1).5, 1, 2, and $k=25(5)40(10)60$, 75, 200, 500, ∞. A somewhat better sequence in Table 2 facilitating harmonic interpolation would have been $k=30$, 40, 60, 120, ∞. *L. A. Aroian* (Culver City, Calif.).

Fraser, D. A. S. Confidence bounds for a set of means. Ann. Math. Statistics 23, 575–585 (1952).

Given n ($n>1$) independent normal variates with unknown means and known common variance. The possibility of one-sided and two-sided confidence intervals for the set of means with confidence coefficient β ($0<\beta<1$) is investigated. The author proves the nonexistence of intervals satisfying mild regularity conditions which have exact confidence level β. However, intervals are given with condence level at least β. *S. W. Nash* (Vancouver, B. C.).

van IJzeren, J. Elementary proof of the independence of mean and variance of samples from a normal distribution. Statistica, Rijswijk 6, 113–119 (1952). (Dutch. English summary)

The proof proceeds in four stages: (1) If x and y are normal, equal and independent variates, so are

$$u=x\cos\theta-y\sin\theta \quad\text{and}\quad v=x\sin\theta+y\cos\theta;$$

(2) hence, $\bar{x}_n=x_n/n+\bar{x}_{n-1}(n-1)/n$ and $x_n-\bar{x}_{n-1}$ are normal and independent; (3) but $\bar{x}_2=\frac{1}{2}(x_1+x_2)$ and $s_2=|x_2-x_1|/\sqrt{2}$ are normal and independent; (4) hence, by induction, $\bar{x}_n$, $x_n-\bar{x}_{n-1}$, s_{n-1} and $ns_n^2=(n-1)s^2_{n-1}+(x_n-\bar{x}_{n-1})^2(n-1)/n$ are independent [cf. Kruskal, Amer. Math. Monthly 53, 435–438 (1946); these Rev. 8, 161]. *H. L. Seal.*

Krooth, Robert S. The sampling variances of some statistics used in univariate discrimination. Ann. Eugenics 17, 302–306 (1953).

Alfred, F. M.-. Modèles de courbes statistiques gouvernant le hasard. Distribution hypergéométrique et système Pearsonien. Hermès, Québec 2, 56–80 (1952).

Expository article. *L. A. Aroian.*

Des Raj. On a generalised Bessel function population. Ganita 3, 111–115 (1953).

The author finds the characteristic function, the cumulants, the first four central moments, and shows how to fit the probability function $f(x)$ by the method of moments, where

$$f(x)=x^{\lambda}\varphi_{\lambda}{}^{\mu}(gx^{\mu})\alpha^{\lambda+1}\exp-\left(\frac{g}{\alpha^{\mu}}+\alpha x\right),\quad 0\leq x<\infty,$$

and

$$\varphi_{\lambda}{}^{\mu}(gx^{\mu})=\sum_{r=0}^{\infty}\frac{(gx^{\mu})^r}{r!\Gamma(1+\lambda+\mu r)},\quad \mu\geq -1,\quad \alpha>0.$$

A numerical example illustrates the method.

L. A. Aroian (Culver City, Calif.).

Watanabe, Yoshikatsu. On the ω^2 distribution. J. Gakugei Coll. Tokushima Univ. 2, 21–30 (1952).

Let $x_1, x_2, \cdots, x_N$ be a sample from the uniform distribution between 0 and 1 and let m_k be the number of sample points falling in the interval $(k-1)/n\leq x\leq k/n$ ($k=1,2,\cdots,n$). The author finds the characteristic function of the limiting distribution (as $N\to\infty$) of

$$\omega^2_{N,n}=\frac{1}{Nn}\sum_{l=1}^{n-1}\sum_{k=1}^{l}\left(m_k-\frac{N}{n}\right)^2.$$

This is the form that Smirnov's modification of von Mises ω^2 statistic takes when the sample observations are grouped. The statistic is distribution free so that the restriction that the x's have a uniform distribution is made without loss of generality.

The formal inversion of the characteristic function is outlined for finite n. The distribution function of

$$\omega_\infty^2=\lim_{n\to\infty}\lim_{N\to\infty}\omega^2_{N,n}$$

is obtained and manipulated into a form suitable for computing. A table of the distribution is given. Though the method used is essentially the same as that used by Smirnov [Mat. Sbornik N.S. 2(44), 973–993 (1937)], some slight modifications introduce some simplifications. An inverse table of this limiting distribution has been published since this article was submitted [Anderson and Darling, Ann. Math. Statistics 23, 193–212 (1952); these Rev. 14, 298]. Anderson and Darling also gave another derivation of the distribution of ω_∞^2 in this latter paper. *D. G. Chapman.*

Masuyama, Motosaburo. The exact distribution of Geary's statistic and its generalization. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 2, no. 1, 1–3 (1952).

Geary [J. Roy. Statist. Soc. (N.S.) 93, 442–446 (1930)] showed that $t=(m_1-m_2Z)/(\sigma_1^2+\sigma_2^2Z^2)^{1/2}$, where $Z=x_1/x_2$, is approximately normally distributed for m_2/σ_2 large and x_1, x_2 distributed normally with means m_1, m_2 and variances σ_1^2, σ_2^2, respectively. The exact distribution of t is derived. The distribution of the generalization of t,

$$t^2=\{a^2y^2-(a\cdot y)^2\}/y^2,$$

$x_i=y_i\sigma_i$, $m_i=a_i\sigma_i$, a and y vectors with components y_i, a_i, is also derived under the assumption that $a\neq 0$.

L. A. Aroian (Culver City, Calif.).

Albert, G. E., and Nelson Lewis. Contributions to the statistical theory of counter data. Ann. Math. Statistics 24, 9–22 (1953).

Il est classique [cf. W. Feller, Courant Anniversary Volume, Interscience, New York, 1948, pp. 105–115; ces Rev. 9, 294] de distinguer deux modèles de compteurs Geiger-Müller: les types I et II; les compteurs réels semblent en fait appartenir à un type intermédiaire; les auteurs définissent un modèle contenant les types I et II comme cas particuliers extrêmes; il serait trop long de rapporter ici cette définition; elle comporte que les temps séparant deux instants d'enregistrement consécutifs sont des variables aléatoires de même loi et mutuellement indépendantes; m étant la densité moyenne des événements à enregistrer, les

auteurs étudient la détermination d'intervalles de confiance pour l'estimation de m à partir des enregistrements.
R. Fortet (Paris).

Craig, C. C. Combination of neighboring cells in contingency tables. J. Amer. Statist. Assoc. **48**, 104–112 (1953).

In computing χ^2 for an $r\times c$ contingency table when some expectations are small, it may be desirable to combine the observations in certain cells. A general method is given for finding the maximum likelihood estimates of the marginal probabilities $p_{i.}$ and $p_{.j}$, and hence for computing χ^2. Explicit solutions are presented for combining a pair of cells in the same row (or column), a block of cells, two pairs which form a 2×2 table, and two row pairs whose columns do not overlap. *W. G. Cochran* (Baltimore, Md.).

Ammeter, Hans. Ein neues Testverfahren für geordnete Beobachtungsreihen und seine mathematischen Grundlagen. Mitt. Verein. Schweiz. Versich.-Math. **51**, 21–36 (1951).

Ammeter, Hans. Der doppelseitige und die einseitigen $(I\chi)^2$-Tests und ihre Leistungsfähigkeit für die wahrscheinlichkeitstheoretische Überprüfung von Sterbetafeln. Bl. Deutsch. Ges. Versicherungsmath. **1**, no. 4, 39–60 (1953).

Given n independent deviations from a hypothetical trend-line observed at equal intervals and each supposedly distributed normally about zero with unit standard deviation, it is desired to test the appropriateness of the trend-line. The author proposes the three alternative criteria

$$(I\chi)_1^2=\frac{2}{n(n+1)}\sum_{s=1}^{n}\left(\sum_{r=1}^{s}x_r\right)^2,\quad (I\chi)_2^2=\frac{2}{n(n+1)}\sum_{s=1}^{n}\left(\sum_{r=s}^{n}x_r\right)^2$$

$$(I\chi)^2=\tfrac{1}{2}[(I\chi)_1^2+(I\chi)_2^2]$$

$$=\frac{1}{n(n+1)}\sum_{s=1}^{n}\sum_{r=1}^{n}(n+1-|r-s|)x_rx_s,$$

where x_r $(r=1, 2, \cdots, n)$ are the n observed variates. He obtains the characteristic functions of all three criteria and provides asymptotic expansions for their frequency distributions. He also considers their respective power functions in two specific examples based on the graduation of the 1939/44 Swiss male population mortality table. The second paper "covers" the first p.p. *H. L. Seal* (New York, N. Y.).

Geary, R. C. Non-linear functional relationship between two variables when one variable is controlled. J. Amer. Statist. Assoc. **48**, 94–103 (1953).

It is a classic proposition that the linear regression of y on x is biased if both variates are subject to measurement error. As noted by J. Berkson [same J. **45**, 164–180 (1950)] the bias is absent if the x-values are "controlled" in the sense that they are allocated to predetermined values, say X, so that the error in x becomes independent of X. Geary discusses Berkson's approach in the case of polynomial regression. [Reviewer's note: A regression residual will usually have to be explained not only by measurement errors but also, and primarily, by causal factors not included among the independent variate(s); if such disturbance factors are uncorrelated with the independent variate(s), the regression will still be unbiased.] *H. Wold* (Uppsala).

Wolfowitz, J. Consistent estimators of the parameters of a linear structural relation. Skand. Aktuarietidskr. **35** (1952), 132–151 (1953).

Let α and β be arbitrary constants. Let $(\xi, \eta=\alpha+\beta\xi)$ and (u, v) be two independent pairs of unobservable random variables and, finally, let $X=\xi+u$ and $Y=\eta+v$ be observable random variables. The paper is concerned with consistent estimates of α and β based on an increasing number n of independent pairs (x_i, y_i), $i=1, 2, \cdots, n$, of observations on X and Y. In the previous treatment of the problem [Neyman, Ann. Math. Statistics **22**, 497–512 (1951); these Rev. **13**, 481] it was assumed (i) at least one of the variables ξ and η is nonnormal and (ii) the nonnormal components, say u_1 and v_1, of "errors" u and v are mutually independent and independent of $u-u_1$ and $v-v_1$, respectively, but otherwise arbitrary. In the paper under review the condition (ii) is replaced by a narrower one (iii) that the pair (u, v) is jointly normally distributed. Furthermore, it is assumed $E(u)=E(v)=0$. Condition (iii) is sufficient for the identifiability of α and of the two cases $\beta=0$ and $\beta=\infty$ which under (ii) are not identifiable. The construction of consistent estimates of α and β is based on the fact that the random variable $Y-a-bX$ is normally distributed about zero if and only if $a=\alpha$ and $b=\beta$. The method is applicable in the multi-dimensional case, however, not to the most general case where the parameters are all consistently estimable. In this respect the author's result is somewhat narrower than that of T. A. Jeeves [ibid. **23**, 146–147 (1952)]. Also it is applicable when conditions (i) and (iii) are replaced by other hypotheses that are sufficiently specific to distinguish between the distribution of $Y-\alpha-\beta X=v-\beta u$ and of $Y-a-bX$ where either $a\neq\alpha$ or $b\neq\beta$ or both. In the last section of the paper, the conditions are restricted still further by assuming that the cumulant of ξ of order $k\geq 3$ exists and differs from zero. The solution given here follows the lines of R. C. Geary [Proc. Roy. Irish Acad. Sect. A. **47**, 63–76 (1942); these Rev. **4**, 21]. *J. Neyman.*

Takashima, Hikochio. Statistical inference for random processes. Sûgaku **4**, 161–172 (1952). (Japanese)

Expository paper dealing mainly with the theory developed by U. Grenander [Ark. Mat. **1**, 67–70 (1949); 195–277 (1950); these Rev. **11**, 376; **12**, 511].
K. Yosida (Osaka).

Goodman, Leo A. Sequential sampling tagging for population size problems. Ann. Math. Statistics **24**, 56–69 (1953).

Given a population, P, of unknown finite size N, with some subsets of known size, a method is given to estimate N on the basis of sequential sampling. The sequential procedure is as follows: given a bounded sequence of positive integers $\{n_i\}$, samples of size n_i are drawn successively; before returning the sample to the population the elements are tagged so that they may be distinguished in successive samplings; sampling is discontinued when a total of L tagged elements, or elements from known subsets, are recovered.

It is shown that a minimum variance unbiased estimate of N can be defined recursively; furthermore, this minimum variance unbiased estimate can be expressed as the ratio of two determinants. Furthermore, if t is the total number of elements sampled, then t^2/N has, in the limit as N tends to infinity, a χ^2 distribution with $2L$ degrees of freedom. From this result, an asymptotic estimate, a confidence interval, and tests for N are derived. A numerical compari-

son suggests that the asymptotic results are adequate for most practical values of N. *D. G. Chapman.*

Hoeffding, Wassily. A lower bound for the average sample number of a sequential test. Ann. Math. Statistics **24**, 127–130 (1953).

An analogue of Wald's inequality [same Ann. **16**, 117–186 (1945), especially p. 156; these Rev. **7**, 131] for the average sample number when testing between two simple hypotheses is given for general sequential procedures for testing between two composite hypotheses. *J. Kiefer* (Ithaca, N. Y.).

Chapman, Douglas G. Inverse, multiple and sequential sample censuses. Biometrics **8**, 286–306 (1952).

A finite population of unknown size N is assumed to be partially marked $k(\geqq 1)$ times and sampled at random after each marking, the $(i+1)$th marking being made on the δ_i non-marked members of the ith random sample, and each sample except the last being returned after marking. On the basis of the sample size n_i, $s_i = n_i - \delta_i$, and t_i = the number of marked individuals in the population when the ith sample is taken, $i = 1, \cdots, k$, the author constructs point and interval estimates and tests for N, assuming sampling to be either direct (with the n_i fixed) or inverse (with the s_i or the δ_i fixed). In the direct case the author, using the sufficiency concept, finds knowledge of the number of individuals in the ith sample that have been marked at the jth marking of no value in constructing estimates of N or corresponding tests. Such data are instead found useful for testing randomness of sampling or constancy of N. The author also constructs a sequential marking and sampling procedure, based on Wald's theory, for testing $N \leqq N_0$ against alternatives $N > N_0$, with given power at an alternative N_1.

D. M. Sandelius (Uppsala).

Jensen, Arne. A short remark on the theory of random sampling and the theory of variance. Skand. Aktuarietidskr. **35** (1952), 195–200 (1953).

The author points out the difficulty introduced into sample survey theory by a refusal to make any assumption about the underlying form of the frequency distribution of the observations obtained on the N sampling units in the finite population. He gives two illustrations of results obtained when such assumptions are made. In the first, exact confidence limits for the population $\bar{Y}$ mean are derived from a simple random sample of size n, the result being

$$\bar{Y} = \bar{y} \pm ts\left(\frac{1-f}{n}\right)^{1/2}$$

where $\bar{y}$, s are the sample mean and S.D., respectively, t has $(n-1)$ d.f. and $f = n/N$.

The second example deals with the linear regression estimate

$$\bar{y}_{lr} = \bar{y} + b(\bar{X} - \bar{x}).$$

By assuming that the finite population is drawn at random from an infinite population in which there is a linear regression of y on x, the author obtains exact confidence limits for $\bar{Y}$ in the form

$$\bar{Y} = \bar{y}_{lr} \pm t's'\left(\frac{1-f}{n} + \frac{(\bar{x}-\bar{X})^2}{\sum(x_i-\bar{x})^2}\right)^{1/2}$$

where t' has $(n-2)$ d.f. and s' is the root-mean-square deviation from the sample regression. Both results are as would be anticipated. *W. G. Cochran* (Baltimore, Md.).

Chakravarti, I. M. Use of the analysis of covariance in two-stage sampling. Calcutta Statist. Assoc. Bull. **4**, 127–129 (1952).

To correct for bias two simple estimates, based on two-stage sampling, for the product of the means of two correlated variates, the author makes use of sample covariances between and within 1st stage sampling units.

D. M. Sandelius (Uppsala).

Tweedie, M. C. K. The estimation of parameters from sequentially sampled data on a discrete distribution. J. Roy. Statist. Soc. Ser. B. **14**, 238–245 (1952).

On a random variable taking N possible values with probabilities depending on parameters $\theta_1, \cdots, \theta_R$ independent observations are made until (*) $\sum_{i=1}^{N} \lambda_i x_i \geqq \zeta$, where ζ (>0) and the λ_i are given constants and x_i is the number of observations taking the ith possible value. (The case $\lambda_1 = \cdots = \lambda_N = 1$ corresponds to sampling with fixed sample size; the case when some of the λ_i equal 1 and the others 0 corresponds to inverse binomial sampling.) By an extension of Wald's fundamental equality, formulas are derived for the means and moment-matrix of the x_i, assuming ζ large. Equations for ML estimates of the θ_j are given. The moment-matrix of these estimates is given implicitly. The problem of choosing the constants of (*) is discussed. Some examples are given. *D. M. Sandelius* (Uppsala).

Horvitz, D. G., and Thompson, D. J. A generalization of sampling without replacement from a finite universe. J. Amer. Statist. Assoc. **47**, 663–685 (1952).

The probabilities with which the several members of a finite population are selected for a sample are allowed to differ and to depend on what items have already been drawn, subject to the condition that the probability that any two items appear in the sample together is predetermined. The authors find the best linear unbiased estimator of the population total for the variate observed, its variance, and an unbiased estimate of this variance. The scheme is extended to two-stage sampling. The remainder of the paper discusses how use can be made of information about a supplementary variable, and how probabilities of selection can be assigned to increase the efficiency of estimation. *S. W. Nash.*

Singh, Baikunth Nath. Use of complex Markoff's chain in testing randomness. J. Indian Soc. Agric. Statistics **4**, 145–148 (1952).

The author gives the asymptotic and (for small m) exact distribution of a statistic which he suggests (but does not specify how) be used to test the independence of a set of m identically distributed 0-or-1 random variables against the alternative that the distribution of each depends on the past only through its two predecessors. He seems unaware that the statistic in question is a simple function of the numbers of runs of various lengths. *J. Kiefer.*

Goodman, Leo A. Serial number analysis. J. Amer. Statist. Assoc. **47**, 622–634 (1952).

A discrete, finite population is numbered serially, then sampled randomly without replacement. The minimum variance unbiased estimator of the size of the population is found, with an unbiased estimator of its variance and an unbiased estimator of the variance of this estimated variance. Tests of hypotheses and confidence intervals are also considered. The estimator of population size is compared with other estimators, in particular Geary's "closest"

estimator. The theory is applied to testing tables of permutations of the first 9 and 16 integers for randomness.

S. W. Nash (Vancouver, B. C.).

Chandler, K. N. The distribution and frequency of record values. J. Roy. Statist. Soc. Ser. B. 14, 220–228 (1952).

The members of a semi-infinite time series are independently and identically distributed with the same density function. A variate is called a lower record value if it is less than all previous variates of the series. The distribution of the rth lower record value is obtained and its probability points for $r=2, 3, \cdots, 9$ and for normal rectangular universes are tabulated. The distribution of the serial numbers of the rth record value and the distribution of the difference of the serial numbers of successive record values are obtained and found to be independent of the original distribution. Their distributions are tabulated for $r=2, 3, \cdots, 9$.

S. W. Nash (Vancouver, B. C.).

Hartley, H. O. Second order autoregressive schemes with time-trending coefficients. J. Roy. Statist. Soc. Ser. B. 14, 229–233 (1952).

In the undamped oscillator $d^2u/dt^2+bu=\epsilon_t$ a trend is introduced in b, or $b=b_0+b_1t$. The solution u_t and its average correlogram as defined by $P(k)=C(k)/C(0)$ with $C(k)=E(\int_{t_0+k}^{t_1} u_tu_{t-k}dt)/(t_1-t_0-k)$ are deduced in terms of integrals for which numerical tables are available.

H. Wold (Uppsala).

Breny, H. Quelques considérations sur la théorie statistique des faisceaux de fibres. Math. Centrum Amsterdam. Rapport S 96, 11 pp. (1952).

Consider a perfectly regular cylindrical thread of length L consisting of a bundle of parallel fibers with the probability function $f(l)$ for the length l of the fibers, and N the average number of heads of fibers per unit length of the thread. For L large the author investigates the random variable $n(t)$, the number of heads of fibers at the point $x=t$ of the thread, and considers in detail the Markov process associated with $n(t)$ under certain assumptions, and also the minimum of $n(t)$ on an interval under the assumption of a Markov process.

L. A. Aroian.

Smith, Nicholas M., Jr., Walters, Stanley S., Brooks, Franklin C., and Blackwell, David H. The theory of value and the science of decision, a summary. J. Operations Res. Soc. Amer. 1, 103–113 (1953).

Geidel, Hans. Zur Verrechnung und Zusammenfassung von Versuchsergebnissen mit der Varianzanalyse. Mitteilungsblatt Math. Statist. 5, 44–51 (1953).

Thawani, V. D. A simple method of construction of symmetrical confounded factorial designs. J. Indian Soc. Agric. Statistics 4, 124–136 (1952).

This is an exposition of R. A. Fisher's methods for the construction of the designs in the title.

H. B. Mann.

Mathematical Economics

Vajda, S. Spieltheorie und statistische Entscheidungsverfahren. Mitteilungsblatt Math. Statist. 5, 52–61 (1953).

Expository paper.

Bellman, Richard. On games involving bluffing. Rend. Circ. Mat. Palermo (2) 1, 139–156 (1952).

The author describes, discusses, and solves some more examples of games of the continuous poker variety and calls attention again to the fact that these games possess pure optimal strategies.

D. Gale (Copenhagen).

Nash, John. Two-person cooperative games. Econometrica 21, 128–140 (1953).

The situation considered is the following. Associated with two players, 1 and 2, are sets S_1 and S_2 of strategies which are compact convex sets (presumably subsets of some topological linear space) and bilinear pay-off functions P_1 and P_2 defined on $S_1\times S_2$, the interpretation being the usual one of game theory. In addition, there is given a compact convex set B in the plane such that if $s_1 \epsilon S_1$ and $s_2 \epsilon S_2$, then $[P_1(s_1, s_2), P_2(s_1, s_2)] \epsilon B$. The points (u_1, u_2) of B are to be interpreted as pairs of pay-offs, or utilities, which can be obtained by the two players either by employing strategies or by some mode of cooperation. The author now defines values v_1 and v_2 for this situation by what amounts to defining a new game whose strategies are still S_1 and S_2 but with new pay-off functions q_1 and q_2, where $[q_1(s_1, s_2), q_2(s_1, s_2)]$ is defined to be the (generally unique) point (u_1, u_2) of B such that $u_i \geqq p_i(s_1, s_2)$ and $(u_1-p_1(s_1, s_2))(u_2-p_2(s_1, s_2))$ is a maximum (this definition must be modified for certain degenerate cases). It is then asserted that this new game has an equilibrium point (s_{10}, s_{20}) which further satisfies the max-min condition, i.e.,

$$v_1=q_1(s_{10}, s_{20})=\max_{s_1}\min_{s_2} q_1(s_1, s_2),$$

$$v_2=q_2(s_{10}, s_{20})=\max_{s_2}\min_{s_1} q_2(s_1, s_2).$$

These numbers v_1 and v_2 are defined to be the desired values.

The bulk of the paper is devoted to justifying the definition of value, or rather the new pay-off functions q_1 and q_2 which lead to the value. This is done from two points of view. First, a negotiation model is set up, and it is argued using the author's previous work [Econometrica 18, 155–162 (1950); Ann. of Math. (2) 54, 286–295 (1951); these Rev. 12, 40; 13, 261] that optimal negotiation will indeed lead to the values v_1 and v_2. Secondly, the author sets up some axioms which he requires any notion of value to satisfy and then indicates that the value obtained above is the only one fulfilling these requirements.

The paper contains many novel and provocative ideas. It is written in the style of an expository note with some indications of proofs, but most of the mathematics must be taken on faith. A more serious failing is the complete absence of worked out examples which one expects to find in works of this sort.

D. Gale (Providence, R. I.).

May, Kenneth O. A note on the complete independence of the conditions for simple majority decision. Econometrica 21, 172–173 (1953).

The author has previously given a set of four conditions which characterize the method of majority decision from among the possible group decision methods for choosing between two alternatives [Econometrica 20, 680–684 (1952); these Rev. 14, 392]. He now shows that these four conditions are completely independent; that is, for any subset of the conditions there is a decision method which satisfies the conditions in this subset but fails to satisfy those in its complement.

D. Gale (Providence, R. I.).

***Dresher, M., and Karlin, S. Solutions of convex games as fixed-points.** Contributions to the theory of games, vol. 2, pp. 75–86. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

This paper is devoted to zero-sum two-person games where each player chooses a point in an arbitrary convex set. The principal results are the following theorems. (A) If a polyhedral game has a unique solution, then the two optimal strategies lie in polyhedra of the same dimension. (B) Suppose that in a polyhedral game, the sets of optimal strategies for players I and II are, respectively, k- and l-dimensional, and are interior to polyhedra which are, respectively, u- and v-dimensional. Then

$$u-k=v-l.$$

The authors show how to symmetrize a finite convex game, and compute the solutions of two illustrative games.

J. Wolfowitz (Ithaca, N. Y.).

***Milnor, John. Sums of positional games.** Contributions to the theory of games, vol. 2, pp. 291–301. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Many common games, such as chess and checkers, exhibit a structure in addition to that which is necessary to define their game-theoretic properties. By a position in such a game will be meant merely the physical setup of the board without any specification as to which player is to move. In these games a set of possible moves is defined at each position for each of the players, even though only one of them will actually be able to move from this position in any particular play of the game. In this paper an operation of addition will be defined and studied for games having this structure. (From the author's introduction.)

J. Wolfowitz (Ithaca, N. Y.).

***Bott, Raoul. Symmetric solutions to majority games.** Contributions to the theory of games, vol. 2, pp. 319–323. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

Let I denote the set of players in an n-person game. If $S\subset I$, let $|S|$ denote the number of players in S. An (n, k)-game is an n-person game whose characteristic function v is given by

$$v(S)=\begin{matrix} -|S| & \text{for} & S<k \\ n-|S| & \text{for} & S\geqq k \end{matrix}$$

where $n/2<k<n$. The author shows that an (n, k)-game has a unique symmetric solution which may be described as follows. Let $p=n-k+1$ and partition I into s sets of p players each and one set of r players, $r<p$. An imputation is in the solution if it assigns equal amounts to all members of a given set of the partition, assigning -1 to each of the r left-over players. The solution consists of the union of these imputations for all such partitions.

D. Gale.

***Gillies, D. B. Discriminatory and bargaining solutions to a class of symmetric n-person games.** Contributions to the theory of games, vol. 2, pp. 325–342. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The author considers (n, k)-games introduced by Bott (see the preceding review), but whereas Bott shows that such games possess unique symmetric solutions, it is here shown that non-symmetric solutions occur in great abundance. Some of the solutions obtained display new types of discriminatory and bargaining behavior.

D. Gale.

***Shapley, L. S. A value for n-person games.** Contributions to the theory of games, vol. 2, pp. 307–317. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

An n-person game is here identified with its characteristic function v, a superadditive real-valued function defined on subsets S of a universe of players U, such that $v(\phi)=0$ and $v(S)\geqq v(S\cap T)+v(S-T)$ for all S, T, contained in U. A subset N of U is called a carrier of v if $v(S)=v(N\cap S)$ for all $S\subset U$. The sum of games on the set U is defined as the sum of the characteristic functions and is again a game. If π is a permutation of U, the game πv is defined by $\pi v(S)=v(\pi^{-1}S)$.

A value $\varphi[v]$ for a game v is a function which associates to each $i\in U$ a real number $\varphi_i[v]$ such that: (1) for any permutation π of U, $\varphi_{\pi i}[\pi v]=\varphi_i[v]$; (2) for each carrier N of v, $\sum_{i\in N}\varphi_i[v]=v(N)$; (3) for any games v and w, $\varphi[v+w]=\varphi[v]+\varphi[w]$. The game-theoretic interpretation of the three conditions shows that they correspond to natural requirements for the notion of a value. Theorem: For games v with a finite carrier, there exists a unique value $[v]$ given by the formula

$$\varphi_i[v]=\sum_{S\subset N}(s-1)!(n-s)!/n![v(S)-v(S-(i))]$$

where s and n are the number of elements in S and N respectively and N is any carrier of v. The author derives further elementary properties of the value function and calculates values for a number of examples. In the final section he describes a bargaining procedure which leads again to the desired value function.

D. Gale.

***Shapley, L. S. Quota solutions of n-person games.** Contributions to the theory of games, vol. 2, pp. 343–359. Annals of Mathematics Studies, no. 28. Princeton University Press, Princeton, N. J., 1953. $4.00.

The set of players in an n-person game is denoted by I, its elements by $i, j, \cdots, m$, etc., its subsets by $\{i, j\}$, $\{k, l, m\}$, etc., and its characteristic function by v. The game v possesses a quota if there is an n-vector $\omega=(\omega_1, \omega_2, \cdots, \omega_n)$ such that (1) $\sum_{i=1}^n\omega_i=v(I)$ and (2) $\omega_i+\omega_j=v(\{i, j\})$, for all $i\neq j$. Games possessing a quota are called quota games and the set of these is denoted by Q. It is shown that all constant-sum four-person games belong to Q and that quota games occur as lower dimensional subsets in the sets of all n-person games, the dimension being a substantial fraction of the full dimension. The author exhibits families of solutions, in the sense of von Neumann, for all quota games, the solutions being typically sets of n line segments joined at a point, or $n-1$ disconnected line segments. The final sections describe more complicated solutions to games in Q, and some generalizations of earlier results to a wider class of games.

D. Gale (Copenhagen).

Gallego-Diaz, José. Functional classification of utility. Revista de Economia, Lisboa 4, no. 2, 57–62 (1951). (Spanish)

The author begins with a straightforward proof of the known proposition that under certain regularity conditions if a function f of the positive vector x satisfies $f(kx)=\varphi(k)f(x)$ identically in the scalar k, then f is actually a homogeneous function of its arguments. Further, if f satisfies $f(k_1x_1, \cdots, k_nx_n)=\varphi(k_1, \cdots, k_n)f(x_1, \cdots, x_n)$, then f is of the form $c\prod_1^n x_i^{m_i}$. It is then asserted, apparently by

analogy with mechanics, that all econometric formulas, particularly utility functions, ought to be of this type. Actually there seems to be little reason to place weight on this kind of dimensional invariance, except of course in things like demand equations, where prices and quantities are involved, and the resulting homogeneity is well-known. But the property in question is not preserved under monotonic transformation of the utility function. Nor do the restrictions implied for production relations seem justifiable.
R. Solow (Cambridge, Mass.).

Gallego-Diaz, José. Riemann spaces and mathematical economics. Revista de Economia, Lisboa 4, 129–140 (1951). (Spanish)

The author considers a $(2n+1)$-dimensional "economic space" (n commodities x_i, n prices y_i, and time) on which he defines a metric by the differential quadratic form

$$ds^2=\sum\frac{dx_i}{x_i}\frac{dy_i}{y_i}+\frac{dt^2}{t^2}.$$

This choice is justified by an appeal to intuition and the analogy of special relativity. In terms of this metric the author proposes a principle of "least economic action": the path between any two points in "economic space" is such as to minimize $\int U ds$ along the path, where U is a utility function (involving prices!), chosen as linear-logarithmic [see preceding review]. The Euler equation of this variational problem is studied, and for t constant, $n=1$, it yields $y^{2N+1}=c_1x^{2N+1}+c_2$, etc. *R. Solow* (Cambridge, Mass.).

Gallego-Diaz, José. The partial differential equations of utility. Revista de Economia, Lisboa 4, 73–86 (1951). (Spanish)

To the conclusions described in the two previous reviews the author adds the assumption that the utility function of a single consumer satisfies the classical partial differential equation of diffusion; and draws conclusions about indifference surfaces, etc. Since these surfaces contain prices as well as quantities as variables, they would apparently have to be interpreted as describing a sort of reservation-demand phenomenon, rather than as conventional indifference surfaces. *R. Solow* (Cambridge, Mass.).

Mathematical Biology

✱Li, Ching Chun. An introduction to population genetics. National Peking University Press, Peiping, 1948. v+321 pp.

The student of population genetics too frequently thinks of all populations as being indefinitely large. Inevitably that is the assumption in most of this book, and it is a good idea, in order to stress the contrary point of view, that it should both begin and end with small populations. Between, one finds expounded with on the whole admirable clarity all the expected topics, in connection with simple mendelian inheritance, multiple alleles, sex-linkage, autopolyploidy, linkage, systems of mating and the technique of path coefficients, with discussion of the effects of mutation, selection and migration in the determination of stationary gene distributions, and their relevance to evolution. The later chapters giving this discussion include some fairly recent work, and follow mainly Sewall Wright's analysis (often very closely).

This book, which has evidently grown out of lecture notes, meets a real need, and mathematicians who have occasion to lecture to genetics students should find it a very useful adjunct to their courses. Knowledge of elementary genetics and some statistical concepts is assumed, but requirements of mathematical competence are kept to a minimum. There are many practical numerical examples and exercises. As far as the reviewer knows it is the only book of its kind, and in view of many virtues its defects may be forgiven, but some must be mentioned. The exposition is not uniformly satisfactory, and is occasionally bad. There are some serious errors (as on pp. 101, 238, 249—contrast p. 275—where the author appears not to know what is meant by stability of equilibrium); and there are many passages which need to be read critically (e.g., the low estimates on pp. 34, 60 of the chance of disproving paternity by blood group technique, which ignore the possibility of combining different tests). Some of the discussions in the later chapters ought to be taken as no more than pointers to further reading. Superficially the book appears to be well documented, but references in the text by author and date only are often ambiguous and more often lacking in the bibliography at the end. Further minor criticisms might be made, but they do not destroy the real merits of the book and it is to be hoped that it will be carefully revised in later editions. *I. M. H. Etherington* (Edinburgh).

Rashevsky, N. Imitative behavior in nonuniformly spatially distributed populations. Bull. Math. Biophys. 15, 63–71 (1953).

The author obtains a system

$$\psi_1'=A_1N_1F_1(\psi_1)+\epsilon_1A_1N_2F_2(\psi_2)-a_1\psi_1,$$
$$\psi_2'=A_2N_2F_2(\psi_2)+\epsilon_2A_2N_1F_1(\psi_1)-a_2\psi_2,$$

where $0\leqq\epsilon_i\leqq 1$, as descriptive of the development of a form of behavior in each of two communities, spatially separated but exerting mutual influence. Most of the paper is concerned with a qualitative study of the solutions.
A. S. Householder (Oak Ridge, Tenn.).

Peyovitch, T. Contribution à l'étude des équations biologiques. Bull. Soc. Math. Phys. Serbie 3, nos. 3–4, 3–10 (1951). (Serbo-Croatian summary)

L'auteur commence par l'équation biologique

$$\frac{dx}{dt}=\alpha+x(\epsilon-lx),$$

ϵ étant coefficient de multiplication, l coefficient limitatif, et α coefficient d'immigration-émigration. Cette équation était déjà étudiée sous une forme plus générale par Kostitzin [Mathematical biology, Harrap, London, 1939, pp. 67–69]. L'auteur continue par le système

$$\frac{dx}{dt}=\alpha_1+\epsilon_1x-l_1xy,\quad\frac{dy}{dt}=\alpha_2+\epsilon_2y-l_2xy,$$

en supposant pour chaque espèce l'absence de la concurrence intraspécifique. Dans son analyse de la stabilité des états stationnaires il ne tient pas compte de ce que ces points ne se trouvent pas nécessairement dans le secteur positif, alors que les variables x, y, exprimant le bilan de deux groupes, sont essentiellement positives. La deuxième substitution (équation 25) n'apporte rien de nouveau: elle transporte tant simplement le début de coordonnées d'un point stationnaire à l'autre. Travail à recommencer.
V. A. Kostitsin (Paris).

Bodenheimer, F. S., and Schiffer, M. Mathematical studies in animal populations. I. A mathematical study of insect parasitism. Acta Bioth. Ser. A. 10, 23–56 (1952).

Les auteurs étudient le parasitisme chez les insectes. En suivant W. R. Thompson ils cherchent à établir les relations entre P (nombre des parasites) et H (nombre des hôtes) d'une génération à l'autre, en admettant un certain nombre du postulats en vue de donner aux équations une forme aussi simple que possible. De temps en temps une "catastrophe" d'origine saisonnière autre extermine presque totalement les hôtes et les parasites. Avec les postulats admis la "catastrophe" frappe les parasites plus sérieusement que les hôtes. Les équations en différences finies, que les auteurs obtiennent, admettent une sommation facile. En remplaçant les variables discontinues par les variables continues on a les équations différentielles

$$(*)\qquad \frac{dH}{dt}=qH-rP,\quad \frac{dP}{dt}=P,$$

avec q et r constants et $z=z_0-z_1x$, x étant la mesure de contamination des hôtes: $x=P/H$. Dans ces conditions x vérifie l'équation logistique

$$(**)\qquad \frac{dx}{dt}=(z_0-q)x-(z_1-r)x^2,$$

dont la solution est connue. Les auteurs étudient le comportement de H, P, suivant les signes de z_0-q, z_1-r. Il considèrent ensuite, en revenant aux différences finies, le cas de superparasitisme et discutent les fluctuations quasi-périodiques.

Je tiens à signaler une affirmation méthodologique qui ne découle nullement de leur étude: "la méthodologie a donné comme résultat intéressant le fait que l'analyse des populations animales peut se faire sans l'application des integrales". En fait, la sommation de leurs équations en différences finies est équivalente à l'intégration et elle n'est facile que grâce aux postulats simplifiants. Tel n'est pas toujours le cas, et très souvent la sommation effective est plus difficile que l'intégration. *V. A. Kostitzin* (Paris).

Hearon, John Z. Nonlinear diffusion in metabolic systems. Bull. Math. Biophys. 15, 15–21 (1953).

The equation $-\nabla\cdot J_k+Q_k=\partial C_k/\partial t$ is considered in conjunction with the simple Fick's law $J_k=-D_k\nabla C_k$, and the general Fick's law $J_k=-\sum D_{kj}\nabla C_j$, allowing the D's and the Q's to depend upon the concentrations. It is assumed that J_k is continuous, and Q_k bounded for $0<r\leqq r_0$ and vanishes elsewhere. Spherical symmetry is presupposed. Examples of the conclusions are: For the simple Fick's law, if Q_k is finite also at $r=0$, then J_k vanishes at $r=0$, and if D_k has a finite limit at $r=0$, then $dC_k/dr\to 0$ at $r=0$. For the general Fick's law, if every Q_k is bounded at $r=0$, then every J_k vanishes at $r=0$, and if every D_{kj} has a limit D^0_{kj} with a non-vanishing determinant $|D^0_{kj}|$, then $dC_k/dr\to 0$ for every k at $r=0$. It is pointed out that in the general case ∇C_k does not vanish with Q_k and that the species may flow against its concentration gradient.

A. S. Householder (Oak Ridge, Tenn.).

Hearon, John Z. Comments on the approximate solution of the diffusion equation. I. Bull. Math. Biophys. 15, 23–31 (1953).

The approximation method of N. Rashevsky is discussed and reviewed. It is shown that in addition to the explicit assumptions and approximations there is involved the assumption that the rate of metabolism is the same at every point in the cell and that the average rate of metabolism is different from zero. An expression is given for the error in the approximate method when the rate of metabolism is any function of the concentration. It is also shown that a solution in the form of that obtained by the approximate method is not possible if the generalized laws of diffusion are assumed to apply. (Author's abstract.)

A. S. Householder (Oak Ridge, Tenn.).

Landahl, H. D. An approximation method for the solution of diffusion and related problems. Bull. Math. Biophys. 15, 49–61 (1953).

Consider a semi-infinite, one-dimensional region, with $c=0$ at $t=0$, $x>0$, but $c=c_0=$const. at $x=0$, $t>0$. The author assumes an approximation of the form $c(x,t)=c_0(1-x/r)$ for $0\leqq x\leqq r$ and $c=0$ for $x>r$, where $r=r(t)$. Then $r(t)$ is determined by the condition that the time integral of the flow across the boundary at $x=0$ is equal to the total amount of solute on the right. In a similar vein the author considers a finite region, a source at $x=0$, a discontinuity in the diffusion coefficient, a two-dimensional flow with cylindrical symmetry, two cases of central symmetry with consumption of the material, giving estimates of errors throughout. *A. S. Householder* (Oak Ridge, Tenn.).

Karreman, George. Some contributions to the mathematical biology of blood circulation. Reflections of pressure waves in the arterial system. Bull. Math. Biophys. 14, 327–350 (1952); erratum, 15, 109 (1953).

Assuming that small pressure waves in an arterial system (incompressible, inviscid fluid in an elastic tube) are of the form $p=f(x\pm ct)$, c denoting the wave velocity, x the distance along the tube, and t the time, the author considers the reflection of such waves at a discontinuity of the tube diameter and (or) tube strength. Noting that partial reflection does not give rise to standing waves, the author questions the claims of some experimenters that standing waves are observed. It is pointed out that by observing the pressure distribution one can obtain information about the degree of reflection occurring at a discontinuity and hence information concerning the condition of the vessels.

In an appendix the author essentially reproduces an analysis by Witzig [Dissertation, Bern, 1914] to determine the dependence of the wave velocity on the viscosity of the fluid. The author, however, includes a number of terms neglected in Witzig's analysis. A result is given which, it is claimed, takes account of two terms neglected by Witzig, which may not be negligible. The reviewer doubts the correctness of this result as it leads to the incredible conclusion that the wave velocity becomes infinitely large for vanishingly small viscosity. The reviewer wonders if errors may have resulted from the fact that the same symbol is used to denote different quantities in the same equation.

G. W. Morgan (Providence, R. I.).

TOPOLOGY

Areškin, G. Ya. On the lattice theory of topological spaces. Akad. Nauk Gruzin. SSR. Trudy Mat. Inst. Razmadze **18**, 53–66 (1951). (Russian. Georgian summary)

This paper contains the proof of a theorem announced earlier [Doklady Akad. Nauk SSSR (N.S.) **81**, 129–132 (1951); these Rev. **13**, 534]. Also discussed are conditions, in terms of bases for closed sets, under which a (bi-)compact T_1-space X admits a continuous mapping onto a (bi-)compact Hausdorff space Y [cf. also Areškin, ibid. (N.S.) **59**, 629–630 (1948); Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 213–220 (1949); these Rev. **9**, 455; **10**, 726].

E. Hewitt (Seattle, Wash.).

Smirnov, Yu. M. On the theory of finally compact spaces. Ukrain. Mat. Žurnal **3**, 52–60 (1951). (Russian)

All topological spaces considered here are Hausdorff spaces. For an infinite, regular cardinal number a, let $\omega(a)$ be the least ordinal number with cardinal number a. A topological space X is said to satisfy condition S_a (S_a') if every decreasing (increasing) transfinite sequence of closed subsets of X, $\{F_\alpha\}_{\alpha<\omega(a)}$, is stationary (i. e., constant from some point onward). The author is concerned with relations existing among the conditions S_a, S_a', and compactness in the interval $[a, \infty]$. [See Smirnov, Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 155–178 (1950); these Rev. **11**, 675.] It will be recalled that $[a, \infty]$-compactness means compactness in the interval $[a, b]$ for all $b \geqq a$, and that an equivalent condition is that every open covering of X admit a subcovering of cardinal number $<a$. The terms "finally compact" and "$[\aleph_1, \infty]$-compact" are synonymous, as are "finally a-compact" and "$[a, \infty]$-compact". Final compactness is the Lindelöf condition referred to by U. S. authors. The author has previously proved [loc. cit., Ch. 2, Th. 2″] that condition S_a is equivalent to hereditary $[a, \infty]$-compactness. The following theorems are proved here. 1. A topological space X satisfies condition S_a' if and only if every subset M of X contains a dense subset of cardinal number $<a$. 2. If X is a locally compact space satisfying both conditions S_a and S_a', then every monotone transfinite sequence $\{B_\alpha\}_{\alpha<\omega(a)}$ of sets B_α which are simultaneously F_σ's and G_δ's is stationary. 3. The Cartesian product of an $[a, b]$-compact space and a (bi-) compact space is $[a, b]$-compact. (Here a need not be regular.) 4. The Cartesian product of a space satisfying condition S_a (S_a') and a space having an open basis of cardinal number $<a$ again satisfies condition S_a (S_a'). 5. Let Q be the real numbers, topologized so that a generic neighborhood of $x \in Q$ is the set $E[y; y \in Q, x \leqq y < x+\delta]$, for all $\delta > 0$. Then Q is hereditarily $[\aleph_1, \infty]$-compact, satisfies conditions $S_{\aleph_1}$ and $S'_{\aleph_1}$, and has the property that $Q \times Q$ is not $[\aleph_1, \infty]$-compact and does not satisfy either condition $S_{\aleph_1}$ or $S'_{\aleph_1}$. The equivalence of conditions $S_{\aleph_1}$ and $S'_{\aleph_1}$ is considered. It is shown that these two conditions are equivalent for ordered topological spaces if and only if the Suslin problem has an affirmative solution; and thus the question is both left unanswered and regarded as very difficult.

E. Hewitt (Seattle, Wash.).

Taimanov, A. D. On multiple separability of closed sets. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 51–62 (1953). (Russian)

The paper begins by proving a special case of a theorem found in Alexandroff and Hopf, Topologie, Bd. I [Springer, Berlin, 1935, p. 71, Satz VI] and a theorem which goes back to Urysohn [Math. Ann. **94**, 262–295 (1925), Satz on page 284]. Among the results new to the reviewer are the following. 1) A topological space is perfectly normal (i.e., every closed set is a G_δ) if and only if for every monotone decreasing sequence $\{F_n\}_{n=1}^\infty$ of closed subsets of X there exists a family of open subsets $\{H_n\}_{n=1}^\infty$ of X such that $H_n \supset F_n$ ($n=1, 2, 3, \cdots$) and $\bigcap_{n=1}^\infty H_n = \bigcap_{n=1}^\infty F_n$. 2) A topological space X is completely normal (i. e., every subspace is normal) if and only if for every finite sequence $F_1, \cdots, F_m$ of closed subsets of X, there exists a sequence of open subsets $H_1, \cdots, H_m$ of X such that $H_i \supset F_i \cap (\bigcap_{j=1}^m F_j)'$ and $\bigcap_{i=1}^m H_i = 0$. 3) If X is a perfectly normal topological space and $\{F_n\}_{n=1}^\infty$ is any sequence of closed subsets of X, then there exists a sequence $\{H_n\}_{n=1}^\infty$ of open subsets of X such that $H_n \supset F_n \cap (\bigcap_{m=1}^\infty F_m)'$ ($n=1, 2, 3, \cdots$) and $\bigcap_{n=1}^\infty H_n = 0$. A number of other, similar, results are also stated and proved.

E. Hewitt (Seattle, Wash.).

Šneĭder, V. E. Continuous images of Suslin and Borel sets. Metrization theorems. Doklady Akad. Nauk SSSR (N.S.) **50**, 77–79 (1945). (Russian)

This is a miscellaneous collection of theorems dealing with topological spaces. We list the following typical examples. 1. A compact Hausdorff space R is metrizable if and only if the diagonal $E[(x, y); (x, y) \in R \times R, x=y]$ is a G_δ in $R \times R$. 2. A compact Hausdorff space which is a continuous image of a Hausdorff space satisfying the second countability axiom is metrizable. 3. Let F_A-sets in a topological space X be the family of Suslin sets [see for example Hausdorff, Mengenlehre, 3rd ed., de Gruyter, 1935, pp. 90–93] obtained from the closed subsets of X. If a completely regular space E is an F_A-set in some compact extension of E, then it is an F_A-set in every compact extension. 4. A topological space X is said to have countable multiplicity if every family $\mathfrak{A}$ of open subsets of X admits a countable subfamily $\mathfrak{B}$ such that $\bigcup_{G \in \mathfrak{B}} G = \bigcup_{G \in \mathfrak{A}} G$. Let E be a Borel set in a space R of countable multiplicity, and let E' be a one-to-one continuous image of E lying in a compact Hausdorff space R' also of countable multiplicity. Then E' is a Borel set in R'.

E. Hewitt (Seattle, Wash.).

Šneĭder, V. E. Descriptive theory of sets in topological spaces. Doklady Akad. Nauk SSSR (N.S.) **50**, 81–83 (1945). (Russian)

Terminology and notation as in the preceding review. The following theorems, among others, are stated; some are proved. 1. Let R be a compact Hausdorff space of countable multiplicity, mapped continuously onto a Hausdorff space containing a subset which is an F_A-set but not a Borel set. Then R also contains a subset which is an F_A-set but not a Borel set. 2. Let E and E' be homeomorphic spaces which are imbedded in spaces R and R', respectively, where R and R' are compact Hausdorff spaces of countable multiplicity. Then: α) if E is an F_A-set, so is E'; β) if the complement of E in R is an F_A-set, so is the complement of E' in R'; γ) if E is a Borel set in R of exactly class ξ, so is E' in R'.

E. Hewitt (Seattle, Wash.).

Sedmak, Viktor. Dimension des ensembles partiellement ordonnés associés aux polygones et polyèdres. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 7, 169–182 (1952). (Serbo-Croatian. French summary)

Let P be a partially ordered set. Then P has a well-defined dimension, as defined by Dushnik and Miller [Amer. J. Math. 63, 600–610 (1941); these Rev. 3, 73]. Note that this dimension is completely unrelated to the dimension of P as defined by G. Birkhoff [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 11; these Rev. 10, 673]. Let Π be a polygon or 3-dimensional polyhedron, and let P be the set whose elements are the vertices, edges, and faces (if Π is a polyhedron) of Π. A partial ordering is defined in P in the natural way, by set-inclusion [see G. Birkhoff, loc. cit., pp. 12–13]. The author proves that the dimension of P is 3 if Π is a polygon and is $\geqq 4$ if Π is a 3-dimensional polyhedron. If Π is one of the regular polyhedra or is a pyramid or prism, then the dimension of P is exactly 4. With an n-dimensional polyhedron Π, there is associated a partially ordered set P in the same way, consisting of all k-dimensional simplices contained in Π ($k=0, 1, 2, \cdots, n$). The dimension of the partially ordered set P in this case is $\geqq n+1$ and is $\leqq$ the number of vertices of Π. For an n-dimensional simplex, the dimension is accordingly equal to $n+1$. The author obtains in this way a partial solution to a problem raised by Kurepa [Teorija skupova, Školska Knjiga, Zagreb, 1951, p. 205, Problem 16.8.1; these Rev. 12, 683]. Kurepa asked for the maximum value of the dimension of P for all 3-dimensional polyhedra Π.

E. Hewitt (Seattle, Wash.).

Moore, R. L. Spirals in the plane. Proc. Nat. Acad. Sci. U. S. A. 39, 207–213 (1953).

The notion of a simple arc α in the plane taking n steps toward spiralling down on a point A inside a circle J for $n=0, 1, 2, \cdots$ is introduced and developed. If, for each $n>0$, α takes n steps toward spiralling down on A, then α is said to spiral down on A. Among other results it is shown that if α spirals down on some point inside J, then the set of all such points is an inner limiting set whose complement is uncountably everywhere dense on α. Also, if M is a compact totally disconnected set inside J, there exists an arc from a point of J which spirals down on each point of M but on no point not in M. An existence theorem is given for a collection G of arcs of the power of the continuum from a point 0 on J, otherwise disjoint, each spiralling down on some point and such that the set of all points on which an arc of G spirals down is a perfect set. *G. T. Whyburn.*

Anderson, R. D. Continuous collections of continuous curves in the plane. Proc. Amer. Math. Soc. 3, 647–657 (1952).

G denotes a nondegenerate collection of disjoint, connected, locally connected continua in the plane. G^* denotes the sum of the elements of G. It is assumed that G^* is compact, and that G is continuous, in the sense that the projection mapping of G^* onto G is not only continuous but interior. A continuous curve is a nondegenerate locally connected continuum. A point p of a continuum M is an emanation point of M if p is the intersection of each two of some three nondegenerate subcontinua of M. The following two theorems furnish a partial characterization of such collections G. (I) If G is connected, and no element of G is degenerate, then G with respect to its elements as points is a hereditarily continuous curve such that the closure of the set of all emanation points of G is totally disconnected. (II) If J is a plane hereditarily continuous curve such that the closure of the set of all emanation points of J is totally disconnected, then there exists a plane continuum M, and a continuous collection of arcs, filling up M, such that G with respect to its elements as points is homeomorphic to J.

Of the additional results, the following are typical. (III) If G is connected, and no element of G is degenerate, then G^* contains a 2-cell. (IV) If G^* is the entire plane, then either every element of G is degenerate or G is a ray whose non-end-points are simple closed curves. *E. E. Moise.*

Hamstrom, Mary-Elizabeth. Concerning continuous collections of continuous curves. Proc. Amer. Math. Soc. 4, 240–243 (1953).

Let G be a collection of disjoint locally connected continua, and let G^* be the sum of the elements of G. It is assumed that G^* is a (compact, metric) continuum, and that G is a continuous collection, in the sense that the projection of G^* onto G is both continuous and interior. A continuum G^* is irreducible if G^* contains points p, q, such that no proper subcontinuum of G^* contains p and q. Theorem 1. If G^* is irreducible, and G is an arc (with respect to its elements as points), then every element of G is degenerate; that is, G^* is itself an arc. (This generalizes a theorem of the reviewer [Bull. Amer. Math. Soc. 55, 810–811 (1949); these Rev. 11, 121] to the effect that such a G could not be a collection of arcs.) Theorem 2. If G^* lies in the plane, and G is an arc, and each element of G is an arc, then G^* is the closure of an open set; and if G^* is also locally connected, then G^* is a topological 2-cell. *E. E. Moise.*

Bing, R. H. Examples and counterexamples. Pi Mu Epsilon J. 1, 311–317 (1 plate) (1953).

Exposition of some topological examples.

Brouwer, L. E. J. Nonconstructible fixed points indicated by classical theorems. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 443–445 (1952). (Dutch)

The author gives a construction which from his point of view defines a topological transformation of a closed square into itself, for which no fixed point can be found, and which cannot have two different fixed points. It is proved that for every topological transformation of the closed square into itself and for every positive number ϵ a point can be found which is displaced by less than ϵ. *A. Heyting.*

de Schwarz, Maria Josepha. Sui principi geometrici del teorema di unicità per le equazioni differenziali ordinarie. Ricerche Mat. 1, 167–184 (1952).

Let C, $\bar{C}$ be rectifiable curves with a common initial point in a Euclidean space. Let s ($0\leqq s\leqq\lambda$) denote the arc length counted from the common initial point, and $P(s)$, $\bar{P}(s)$ the corresponding points on C, $\bar{C}$, respectively. Then the angle $\varphi(s)$ between the tangent vectors to C, $\bar{C}$ at $P(s)$, $\bar{P}(s)$ is defined for almost all s. The author proves (Theorem I): If the ratio of $1-\cos\varphi(s)$ to the square of the distance from $P(s)$ to $\bar{P}(s)$ is bounded for almost all s, then C and $\bar{C}$ coincide. In case that C and $\bar{C}$ are solutions of a system of ordinary differential equations, Theorem I is essentially a reformulation of the well-known uniqueness theorem for solutions of such a system under a Lipschitz condition.

The main part of the paper deals with various generalizations; e.g., it is shown that Theorem I (with properly generalized definitions) is true in the "Tetraeder-Räumen"

of Menger, and that it fails to be generally true for the so-called semi-metric spaces. The answer to the question concerning the validity of the theorem in all metric spaces is left to a later paper. *E. H. Rothe* (Ann Arbor, Mich.).

Kapuano, Isaac. Sur les surfaces homéomorphes à un disque dans un R^3. C. R. Acad. Sci. Paris **236**, 1229–1231 (1953).

An open (closed) calotte is a space homeomorphic with an open (closed) two-cell. This abstract asserts (among other things) the existence of a closed calotte in three-space not contained in any open calotte. *A. D. Wallace.*

Keller, Ott-Heinrich. Zur unmittelbaren Anschaubarkeit 4-dimensionaler Gegenstände: Ein anschaubares singularitätenfreies topologisches Modell der projektiven Ebene im R_4. Math. Nachr. **8**, 179–183 (1952).

A simple closed surface S, homeomorphic to the projective plane, is constructed in Euclidean 4-space in such a way that S consists of a finite number of rectangles with sides parallel to the coordinate axes. This surface is then used to give non-trivial examples illustrating several linking theorems. *W. R. Scott* (Lawrence, Kan.).

Harrold, O. G., Jr., and Moise, E. E. Almost locally polyhedral spheres. Ann. of Math. (2) **57**, 575–578 (1953).

Let K be a topological 2-sphere in Euclidean R^3, let A be the interior of K, and B the exterior of K compactified at infinity by a single point. The authors prove the following. I. If K is locally polyhedral except perhaps at one point, then A and B are simply connected and one of the sets $\bar{A}$ or $\bar{B}$ is a topological 3-cell. II. If K is locally polyhedral except perhaps at 3 points, then one of the sets A or B is simply connected. Examples are known where K is locally polyhedral with the exception of four points and neither A nor B are simply connected. *S. Eilenberg.*

Borovikov, V. On the intersection of a sequence of simplexes. Uspehi Matem. Nauk (N.S.) **7**, no. 6(52), 179–180 (1952). (Russian)

L'intersection d'une suite décroissante de simplexes est un simplexe. Ce théorème qui semble évident, mais ne l'est pas, a été conjecturé par Kolmogorov. L'auteur le démontre. *H. Freudenthal* (Utrecht).

Rado, Tibor. On general cohomology theory. Proc. Amer. Math. Soc. **4**, 244–246 (1953).

The author proposes to define the pth cohomology group of the pair (X, A) in terms of functions $c^p(x_0, \cdots, x_p)$, $x_i \varepsilon X$, which vanish on A and vanish "locally" on X. This replaces the usual procedure of factoring the group of all functions on X vanishing on A by those functions which vanish locally on X, and produces a dimensional discrepancy of one. Thus $H^{p+1}(X, A)$, in the author's sense, is isomorphic to $H^p(X, A)$ (reduced if $p=0$) in the sense of Spanier and Wallace [see Spanier, Ann. of Math. (2) **49**, 407–427 (1948); these Rev. **9**, 523]. The motivation is to minimize the number of identification operations and to define the cohomology groups directly in terms of a suitable cochain complex. *P. J. Hilton* (Cambridge, England).

Fáry, István. Sur les anneaux spectraux de certaines classes d'applications. I. Généralités. C. R. Acad. Sci. Paris **235**, 686–688 (1952).

The purpose of this note is to indicate the terminology, notations, and definitions which the author intends to use in a sequence of notes to follow. No new results are stated. *W. S. Massey* (Providence, R. I.).

Fáry, István. Sur les anneaux spectraux de certaines classes d'applications. II. Fonctions numériques (continues). C. R. Acad. Sci. Paris **235**, 780–782 (1952).

Let X be a topological space, and f a real-valued continuous function defined on X. The author indicates how J. Leray's spectral sequence of a continuous map [cf. J. Math. Pures Appl. (9) **29**, 1–80, 81–139 (1950); these Rev. **12**, 272] can be applied to study the homology properties of the map f, and thus obtain results analogous to those of M. Morse. *W. S. Massey* (Providence, R. I.).

Fáry, István. Sur les anneaux spectraux de certaines classes d'applications. III. Fonctions numériques (différentiables). C. R. Acad. Sci. Paris **235**, 1272–1274 (1952).

The author specializes the situation studied in a previous note [cf. the preceding review] to the case of a real-valued, differentiable function defined on a differentiable manifold. Special attention is given to the study of the behavior of the function in the neighborhood of a critical point. *W. S. Massey* (Providence, R. I.).

Fáry, István. Sur les anneaux spectraux de certaines classes d'applications. IV. Cohomologie des hypersurfaces algébriques. C. R. Acad. Sci. Paris **235**, 1467–1469 (1952).

The purpose of this note is to study the cohomology of algebraic hypersurfaces in complex affine space. The method used is to consider the polynomial which defines the hypersurface as a continuous mapping of complex affine space into the complex plane, and to apply Leray's theory of the spectral sequence of a continuous map to this polynomial. *W. S. Massey* (Providence, R. I.).

GEOMETRY

Thébault, Victor. Sur la géométrie du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **67**, 5–12 (1953).

Sz.-Nagy, Gy. Mittelkante, Mittelebene, Mittelpunkt von Kanten einer Ecke. Publ. Math. Debrecen **2**, 161–165 (1952).

A half-line AE_kA_k ($k=1, 2, \cdots, n$) issued from the fixed point O is said to be an edge (Kante) of the vertex (Ecke) O. The point A_k is an arbitrary point of the edge (distinct from O), and E_k the trace on $0A_k$ of the unit sphere, center O. If S is the centroid of the unit points $E_1, \cdots, E_n$, the line $0S$ is the median edge (Mittellinie) of the n edges; the plane through O perpendicular to OS is the median plane (Mittelebene) of the n edges. The point O is the median point (Mittelpunkt) if S coincides with O, the median line and plane being in such a case indeterminate.

From these basic notions the author derives several propositions, like the following: If four edges of a vertex have a median point, the four unit points are either the ends of two diameters of a circle, or the vertices of an isosceles

tetrahedron. As a consequence of the latter, the author points out that the sum of the squares of the edges of an isosceles tetrahedron is equal to four times the square of the circumdiameter of the tetrahedron [cf. Altshiller-Court, Modern pure solid geometry, Macmillan, New York, 1935, p. 102, ex. 18].

The author concludes by supplying a vector proof of a proposition which Steiner stated without proving [Gesammeite Werke, Bd. II, Reimer, Berlin, 1882, p. 35].

N. A. Court (Norman, Okla.).

Kertész, A., and Szele, T. On the smallest distance of two lines in 3-space. Publ. Math. Debrecen 2, 308–309 (1952).

***Danielsson, Gösta. Proof of the inequality**

$$d^2 \leqq (R+r)(R-3r)$$

for the distance between the centres of the circumscribed and inscribed spheres of a tetrahedron. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 101–105. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let R and r be the radii of the circumscribed and inscribed spheres of a tetrahedron and let d be the distance between their centers. The author proves that $d^2 \leqq (R+r)(R-3r)$.

P. Erdös (Los Angeles, Calif.).

Backes, F. Démonstration élémentaire de la configuration des quinze cercles de Stephanos. Mathesis 62, 12–14 (1953).

Alaime, R. Sur une fermeture hexagonale. Mathesis 62, 30–33 (1953).

Goormaghtigh, R. Sur l'isopôle. Mathesis 62, 34–38 (1953).

Vitale, Darwin Raffaele. L'angolo come grandezza vettoriale autonoma. Ricerca, Napoli 3, no. 3–4, 64–71 (1952).

Giovanardi, Mario. Prospettiva di una superficie di rotazione su un quadro inclinato. Ricerca, Napoli 3, no. 3–4, 25–36 (1952).

***Piazzolla Beloch, M. Geometria descrittiva.** 2a ed. Istituto di Geometria dell'Università di Ferrara, Ferrara, 1953. xv+172+172+118 pp.

This book is based on the course given by the author at the University of Ferrara. It consists of an introduction and ten parts organized into three, separately paginated, sections. Section 1 contains the introduction and parts I and II, section 2 contains part III while section 3 consists of parts IV–X.

In the introduction the author makes some general remarks on various methods of projection and reviews some properties of curves and surfaces as well as some basic concepts of projective geometry. In part I it is shown that an object in space is determined by two perspectives. A construction is given which permits the derivation of a third perspective from a given pair of perspective images. The author's aim is to use this "photogrammetric principle" as the basis for the study of Descriptive Geometry. This approach is essentially equivalent to one used by E. Müller and his school [E. Müller, Vorlesungen über darstellende Geometrie, Bd. I, Deuticke, Leipzig-Wien, 1923; see esp. pp. 124 ff.]. The second part deals with orthogonal projections. The mapping of points, lines, and planes as well as the basic constructions are discussed in great detail. Numerous examples are given and are used to solve graphically elementary problems of solid geometry. Part III deals with applications of the method of orthogonal projection. These are designed to train the student to visualize spatial objects. Conic sections and their orthogonal projections are studied; polyhedra are represented and intersected with planes and also with other polyhedra. A detailed discussion of surfaces of revolution, translation and helical surfaces is followed by a brief chapter on ruled surfaces. The last section deals with a great variety of topics: cartographic projection (part IV), central projection (part V), axonometric projection (part VI), shadows (part VII), application of cartographic projection (part VIII), linear perspective (part IX), photogrammetry (part X). The author calls linear perspective the axonometric representation of an object from a finite center of projection.

The book is well written and contains a great wealth of examples which are carried out in every constructive detail. The use of analytic geometry is almost consistently avoided. The figures though excellently drawn are poorly reproduced. The reason is that the book was prepared from a typewritten manuscript by a rather unsatisfactory lithographic process, very much to the detriment of the appearance of the book. The absence of references to the literature and of an alphabetical index is regrettable. *E. Lukacs.*

Cavallaro, Vincenzo G. Sur l'ellipse de Steiner par les segments torricelliens. Mathesis 62, 21–30 (1953).

Fernández Biarge, Julio. Circular cubics. Gaceta Mat. (1) 4, 238–247 (1952). (Spanish)

Rangaswami Aiyer, K. On a system of circles represented by a Steiner quartic surface. Ganita 3, 103–109 (1952).

Herzog, E. R. Die Anwendung der Riemannschen Zahlenkugel zur Herleitung der sphärisch-trigonometrischen Hauptsätze. Elemente der Math. 8, 31–35 (1953).

Laurenti, F. Sur un système d'équations trigonométriques. Mathesis 62, 9–11 (1953).

Allen, E. F. An extended inversive geometry. Amer. Math. Monthly 60, 233–237 (1953).

Fischer, Helmut Joachim. Geometrische Netze und Konfigurationen und ihre Beziehungen zur Vektorrechnung und Zahlentheorie. I. Collectanea Math. 4, no. 2, 57–119 (1951).

Four coplanar points P_1, P_2, P_3, P_4, no three collinear, can be joined by six lines which have three further intersections; these can be joined to the original points and to one another by further lines, which intersect in further points, and so on. The system of points and lines arising in this manner is called a "net" [Möbius, Der barycentrische Calcül, Barth, Leipzig, 1827, §200]. Any four points of the net (no three collinear) can be used instead of the P's for constructing the whole net [ibid., §204]. If $P_1P_2P_3$ is the triangle of reference, while P_4 is (1, 1, 1), the net consists of all the points and lines whose projective coordinates are integers [ibid., §§201, 203]. The author recapitulates these results, using a different coordinate system which puts the

four P's all on an equal footing. He emphasizes the self-duality of the net by showing that it can be derived just as well from four lines G_i which are related to the four points P_i in a unique manner: the quadrangle $P_1P_2P_3P_4$ and the quadrilateral $G_1G_2G_3G_4$ form together a special Desargues configuration—special in that the point $P_1P_2\cdot P_3P_4$ lies on the line $(G_1\cdot G_2)(G_3\cdot G_4)$, and so on.

His coordinates A_1, A_2, A_3, A_4 (whose sum is zero) are related to Möbius's ϕ, χ, ψ as follows:

$$A_1=3\phi-\chi-\psi,\quad A_2=-\phi+3\chi-\psi,$$
$$A_3=-\phi-\chi+3\psi,\quad A_4=-\phi-\chi-\psi.$$

Each point or line belongs to a family of 3, 4, 6, 12, or 24, derivable from one another by permuting the four coordinates (or the four original points or lines).

He defines the degree (Stufe) of a point or line as follows. The four P's and the four G's are said to be of degree 0. Any other element of the net is said to be of degree n if it is the join or intersection of two elements, one of degree $n-1$ and the other the same or lower. Thus he finds one family of degree 1, two families of degree 2, four of degree 3, seventy-five of degree 4, and he estimates that there must be about one hundred thousand of degree 5.

Apparently a simpler way to achieve the same classification would have been to use ordinary projective coordinates referred to the common diagonal triangle of the quadrangle and quadrilateral. Then the point coordinates of the P's and the line coordinates of the G's would have been 1, ±1, ±1, and the members of a family would have been derivable from one another by permuting the three coordinates and reversing any of their signs. Such coordinates are

$$x=\tfrac{1}{2}(A_2+A_3)=-\phi+\chi+\psi,\quad y=\tfrac{1}{2}(A_3+A_1)=\phi-\chi+\psi,$$
$$z=\tfrac{1}{2}(A_1+A_2)=\phi+\chi-\psi,$$

which would enable us to express joins and intersections by the usual formula for vector multiplication. However, we can expect the author's more complicated coordinates to justify themselves in the promised sequel, on nets in three dimensions. *H. S. M. Coxeter* (Toronto, Ont.).

Herrmann, Horst. Struktureigenschaften, Figurmatrizen und Zergliederungen projektiver Konfigurationen. Math.-Phys. Semesterber. 3, 90–115 (1953).

The author describes several methods for combining and modifying configurations; e.g., he generalizes the procedure whereby a complete quadrangle and quadrilateral are combined to form a Desargues configuration. By removing three skew lines from a complete 6-point in five dimensions, he derives a configuration of six points, twelve lines, eight planes, twelve 3-flats and six 4-flats, having an octahedron for its projection and a cube for its section. Extending the results of an earlier paper [Arch. Math. 2, 207–215 (1950); these Rev. 11, 736], he shows how most of the classical configurations (such as that formed by the 27 lines and 45 tritangent planes of a cubic surface) can be derived by numerical modification of a "binomial" configuration $\binom{n}{k}^{(r)}$, which is an r-flat section of a complete n-point in projective $(k+r-1)$-space. *H. S. M. Coxeter* (Toronto, Ont.).

***Fenchel, W. A generalization of spherical trigonometry.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 139–147. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The classical formula $\cos c=\cos a \cos b$, for a right spherical triangle, has the kinematical counterpart $C^2=A^2B^2$, where A, B, C are rotations of magnitudes a, b, c about the poles of the three sides. When the points of the sphere are interpreted in the complex projective line, the right angle between a and b indicates that the two pairs of fixed points of the collineations A and B are harmonic. Generalizing these notions to complex projective n-space, the author considers two collineations whose fixed points form a pair of harmonically related simplexes, which means that each vertex and opposite hyperplane of either simplex are harmonic pole and polar with respect to the other. Thus, when one simplex is taken as simplex of reference while the other has vertices $(r_{0j}, \cdots, r_{nj})$ $(j=0, \cdots, n)$, then $\sum_{i=0}^{n} r_{ij}r_{ik}^{-1}=0$ $(j\neq k)$. In particular, $\|r_{ij}\|$ may be any Hermitian matrix of roots of unity; e.g., it may be any orthogonal matrix of 1's and -1's. Hence, harmonically related simplexes exist in complex space of any number of dimensions, and in real n-space for $n=1, 3, 7, 11, \cdots, 87$ and presumably also for other values of the form $4t-1$ [see R. E. A. C. Paley, J. Math. Physics 12, 311–320 (1933)]. *H. S. M. Coxeter.*

Klingenberg, Wilhelm. Beziehungen zwischen einigen affinen Schliessungssätzen. Abh. Math. Sem. Univ. Hamburg 18, 120–143 (1952).

The equivalence of a number of affine configuration theorems is shown. In particular, the figure A expressing the associativity of multiplication in Hilbert's calculus of intervals is shown to imply the Theorem of Desargues, and hence to be equivalent to it. By showing that A is a consequence of the Theorem of Pappus, the author also obtains a new proof of the Theorem of Hessenberg. Various equivalences involving the little Theorem of Desargues are given. But he is unable to show that the Reidemeister figure is equivalent to the Theorem of Desargues. This figure is the associative law of multiplication for a plane coordinatized by a ternary ring. *Marshall Hall* (Columbus, Ohio).

Cronheim, Arno. A proof of Hessenberg's theorem. Proc. Amer. Math. Soc. 4, 219–221 (1953).

Der Hessenberg'sche Beweis des Desargues'schen Satz aus dem Satz von Pappus auf Grund der projektiven Verknüpfungssätze bezieht sich auf den Fall perspektiver Dreiecke in allgemeiner Lage, d.h. von den Punkten A_i, B_i $(i=1, 2, 3)$ und S wird angenommen, dass A_i, B_i, S collinear sind, aber A_1, A_2, A_3 ebenso B_1, B_2, B_3 nicht collinear sind, dass die Gerade A_iB_i verschieden ist von der Geraden A_jB_j für $i=j$ und dass sonst keine zusätzlichen Bedingungen für die A_i, B_j bestehen. Verf. schliesst diese Lücke durch folgende Fallunterscheidung: 1. Es gibt eine Permutation (i, j, k) von $(1, 2, 3)$, so dass A_i, B_j, B_k nicht collinear sind und B_k, A_i, A_j nicht collinear sind. In diesem Falle kommen die Überlegungen von Hessenberg zur Anwendung. 2. Es gibt keine Permutation (i, j, k) mit dieser Eigenschaft. Alsdann folgt ohne Beschränkung der Allgemeinheit, dass A_1, B_2, B_3 und A_2, B_1, B_3 und A_3, B_1, B_2 collineare Tripel sind. Die axiale Lage der perspektiven Dreiecke A_1, A_2, A_3 und B_1, B_2, B_3 ergibt sich dann ebenfalls durch dreimalige Anwendung des Satzes von Pappus. *R. Moufang* (Frankfurt a.M.).

Rozenfel'd, B. A. The geometry of a manifold of planes of a projective space as a projective geometry of points. Trudy Sem. Vektor. Tenzor. Analizu 9, 213–222 (1952). (Russian)

Taking real matrices of dimension m as coordinates, the author constructs a projective space P_n^m of dimension n. The space P_{n+1}^{m+1} can be mapped into the space P_{mn+m+n} and their collineation groups are isomorphic. Following Hua

[C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 303–306 (1946); these Rev. **8**, 328] he also discusses a geometry of symplectic matrices. *Marshall Hall* (Columbus, Ohio.).

Convex Domains, Extremal Problems Integral Geometry

Hammer, P. C., and Sobczyk, Andrew. Planar line families. I. Proc. Amer. Math. Soc. **4**, 226–233 (1953).

Dans un plan, une famille F de droites est dite extérieurement simple si elle recouvre une fois et une seule tous les points extérieurs à un cercle de rayon suffisamment grand (y compris les points à l'infini). On montre que toute famille F est celle des diamètres d'une famille d'orbiformes parallèles et inversement. *J. Favard* (Grenoble).

Gale, David. On inscribing n-dimensional sets in a regular n-simplex. Proc. Amer. Math. Soc. **4**, 222–225 (1953).

Jung proved that the smallest (solid) n-sphere of euclidean n-space E_n in which every subset of E_n with diameter 1 may be inscribed has radius $r=[n/2(n+1)]^{1/2}$ [J. Reine Angew. Math. **123**, 241–257 (1901)]. Since the circumscribed sphere of the equilateral n-simplex with unit edge has this radius, Jung's result shows that a sphere of E_n will cover any subset of E_n with diameter 1 whenever it has the obviously necessary property of covering that simplex. In this note the author is concerned with covering subsets of E_n, with diameter 1, by smallest equilateral n-simplices. Clearly such an n-simplex must contain (in fact, be circumscribed about) the n-sphere of diameter 1. The author establishes very simply that this necessary condition is also sufficient. The result is applied to prove for E_2 a sharpened form of Borsuk's conjecture that each subset of E_n, with diameter 1, is the set sum of $n+1$ sets, each of which has diameter less than 1. The bound for the diameters of the three summands in the planar case is $3^{1/2}/2$, which is easily seen to be the best possible. [Reviewer's note. The phrase "less than $3^{1/2}/2$" in the second paragraph of the Introduction should be replaced by "less than or equal to $3^{1/2}/2$". See Theorem II and the Remark following it.]

L. M. Blumenthal (Columbia, Mo.).

Hadwiger, H. Einige neue Ergebnisse über extremale konvexe Rotationskörper. Abh. Math. Sem. Univ. Hamburg **18**, 38–52 (1952).

Let A be a convex solid of revolution in euclidean k-space. Thus the intersections of A with the $(k-1)$-spaces perpendicular to its axis of revolution are $(k-1)$-spheres. Let $a(A)$ denote their maximum radius. Let $W_\lambda(A)$ be the λth mixed volume of A with the unit k-sphere [$\lambda=1, \cdots, k$; thus $W_0(A)=$volume of A; $kW_1(A)=$area of A, etc.]. Let ν be fixed; $0\leqq\nu<k$. Let K and Z be the cone and cylinder, respectively, which satisfy $a(K)=a(Z)=a(A)$ and $W_\nu(K)=W_\nu(Z)=W_\nu(A)$. Then

$$(1)\quad W_\mu(A)\leqq\max[W_\mu(K), W_\mu(Z)] \quad (\mu=\nu+1, \cdots, k).$$

The explicit values of the right-hand terms of (1) are too complicated to be quoted. The proof of (1) is elementary: Let A^n denote convex solids of revolution which satisfy $a(A^n)=a(A)$ and $W_\nu(A^n)=W_\nu(A)$ and which are the sums of altogether not more than n cylinders, cones, and truncated cones. The case $A=A^2$ of (1) is verified by direct computations. On account of known properties of the W_λ's, this readily implies $W_\mu(A^n)\leqq\max W_\mu(A^{n-1})$. Thus (1) holds if A is an A^n. In the general case, A can be uniformly approximated by A^n's. This yields (1). *P. Scherk.*

Sholander, Marlow. On certain minimum problems in the theory of convex curves. Trans. Amer. Math. Soc. **73**, 139–173 (1952).

Let A, C, D, and E denote the area, the perimeter, the diameter, and the thickness of a planar convex body, resp. The paper deals with problems of the following type: Given two of these quantities, to find the extremum values of a third one and to determine the bodies for which the extremum values are attained. Several of these problems have been solved, completely or partially, by various authors. [For the many references and historical details the reader must be referred to the paper itself or to T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934, pp. 80–83 (where, however, the inequalities (3), (5), (6) on p. 81 are stated incorrectly).] For three problems, viz. to find the bodies of minimum A 1) for given D and E [(D, E) problem], 2) for given C and E [(C, E) problem], 3) for given C and D [(C, D) problem], only partial results have been established so far. In the present paper the author solves the (D, E) and (C, E) problems completely and shows that the bodies solving the (C, D) problem must belong to a narrow class of polygons obtainable from Reuleaux polygons in a certain way. A characteristic feature of the three problems in question is that the type of the extremum bodies depends on whether the ratio of the two given quantities does or does not exceed a certain number. Thus, if $E/D\leqq3^{1/2}/2$, the (D, E) problem is solved by triangles; if $1\geqq E/D>3^{1/2}/2$, the solution is the convex hull of an equilateral triangle with side D and three circular arcs with radius E about the vertices. A main step in the author's method is the following. To every planar convex body K there corresponds one and only one body K^*, the centralization of K, which has a centre of symmetry and, in each direction, the same width as K. Centre equivalent bodies, i.e., bodies with identical centralizations, have the same C, D, and E. Thus, for all three problems, the class of competing bodies may be restricted to those solving the minimum area problem within their classes of centre equivalent bodies. Using a method due to Lebesgue, the author shows that these minimum area bodies belong to the class of "triarcs". A triarc is, by definition, a planar convex body whose boundary contains 3 (not necessarily different) points such that at least one line of each pair of parallel supporting lines goes through one (or more) of these points. The further discussion splits up into many details. *W. Fenchel.*

Schütte, K., und van der Waerden, B. L. Das Problem der dreizehn Kugeln. Math. Ann. **125**, 325–334 (1953).

By an ingenious combination of trigonometry and topology, the authors have settled an old question by proving that no more than twelve equal, non-overlapping spheres can touch a single sphere of the same size. Their two proofs supersede the earlier, incomplete proof by A. H. Boerdijk [Philips Research Rep. **7**, 303–313 (1952), pp. 310–312; these Rev. **14**, 310]. *H. S. M. Coxeter* (Toronto, Ont.).

Bambah, R. P., and Davenport, H. The covering of n-dimensional space by spheres. J. London Math. Soc. **27**, 224–229 (1952).

It is proved that the density ϑ of any covering of n-dimensional space by equal spheres (whose centres form a lattice) satisfies $\vartheta>\frac{4}{3}-\epsilon_n$ where $\epsilon_n\to0$ as $n\to\infty$. It is shown that

if Π is a convex polyhedron of volume V with at most $2(2^n-1)$ faces and is inscribed in an n-dimensional sphere of volume J_n and if the foot of the perpendicular from 0 to each face falls inside the face, then $J_n/V>4-\epsilon_n$. This is done by splitting Π into pyramids with apex at the centre of the sphere and considering their volumes. This result is applied to Voronoi's polyhedron Π which consists of those points which are nearer to a given point of the lattice than to any other lattice-point, and yields the inequality stated. A result in the other direction, namely that there exists a lattice-covering for which $\vartheta<(1.15)^n$ has been given by Davenport [Rend. Circ. Mat. Palermo (2) 1, 92–107 (1952); these Rev. 14, 75]. *R. A. Rankin* (Birmingham).

Molnár, J. Ausfüllung und Überdeckung eines konvexen sphärischen Gebietes durch Kreise. I. Publ. Math. Debrecen 2, 266–275 (1952).

L. Fejes Tóth [Comment. Math. Helv. 23, 342–349 (1949); these Rev. 11, 455] proved that the density of the packing of three or more equal small circles on a sphere is less than $\pi/2\sqrt{3}$, and conjectured that this holds not only for the whole spherical surface but also for any convex part of it. The author of the present paper verifies this conjecture. He defines for each circle a kind of Dirichlet region, consisting of all points that are nearer to its centre than to the centre of any other one of the circles. He shows that this spherical polygon need not be allowed to have more than seven sides, nor less than five. He considers, in turn, the heptagon, hexagon, and pentagon, each of which requires some intricate trigonometrical computation.
H. S. M. Coxeter (Toronto, Ont.).

Ohmann, D. Eine Abschätzung für die Dicke bei Überdeckung durch konvexe Körper. J. Reine Angew. Math. 190, 125–128 (1952).

Let K be a convex body in euclidean n-space. Denote by $b(K,\xi)$ and $\Delta(K)$ the width of K in the direction ξ and the minimal width of K, resp. The author asserts: Let K_μ, $\mu=1,2,\cdots,m$, be convex bodies which cover K completely. Then

$$(*)\qquad \sum_{\mu=1}^{m}\frac{b(K_\mu,\xi_\mu)}{b(K,\xi_\mu)}\geq 1$$

for arbitrary directions ξ_μ. Thus, in particular,

$$\sum\Delta(K_\mu)\geq\Delta(K).$$

The last inequality is known. The case $m=n=2$ is due to S. Straszewicz [Ann. Soc. Polon. Math. 21, 90–93 (1948); these Rev. 10, 205] and the general case to Th. Bang [Mat. Tidsskr. B. 1950, 49–53 (1950); these Rev. 12, 352; cf. also Proc. Amer. Math. Soc. 2, 990–993 (1951); these Rev. 13, 769]. For, without restricting the generality, the bodies K_μ may be replaced by parallel strips. (*) has been conjectured by Bang. (Apparently, Bang's paper was not known to the author.) By approximations and affine mapping it is shown that it suffices to prove (*) in the special case where $m=n$, ξ_μ mutually orthogonal, $b(K,\xi_\mu)=1$, and K has interior points. In this case the author constructs a mass distribution over K such that, roughly speaking, for every parallel strip S_μ normal to a ξ_μ the total mass $M(S_\mu\cap K)$ contained in $S_\mu\cap K$ equals the width of S_μ. From the assumption that the strips S_μ, $\mu=1,2,\cdots,n$, cover K the author infers without further argument the inequality $\sum M(S_\mu\cap K)\geq M(K)$ and thus (*). This inequality is, however, not at all obvious since negative masses necessarily occur. *W. Fenchel.*

Ohmann, D. Über die Summe der Inkreisradien bei Überdeckung. Math. Ann. 125, 350–354 (1953).

Der Verf. beweist den folgenden Satz: Wird ein konvexer Bereich der euklidischen Ebene durch eine endliche Anzahl von konvexen Bereichen vollständig überdeckt, so ist die Summe der Inkreisradien der überdeckenden Bereiche nicht geringer als der Inkreisradius des überdeckten Bereichs.
D. Gale (Copenhagen).

Soós, Gy. Ausdehnung des Hellyschen Satzes auf den Fall vollständiger konvexer Flächen. Publ. Math. Debrecen 2, 244–247 (1952).

Let F be the boundary of a convex set with interior points in E^3, but not two parallel planes. A set on F is called convex if it contains with every two points all shortest connections (on F) of the two points. A convex domain on F is a compact convex set with interior points whose boundary consists of a finite number of simple arcs. The following generalization of Helly's theorem is proved: If $K_1, K_2, \cdots$ are convex domains on F and the intersection of any three K_i is not empty, then the intersection of all K_i is not empty.
H. Busemann (Los Angeles, Calif.).

Santaló, L. A. On pairs of convex figures. Gaz. Mat., Lisboa 12, no. 50, 7–10 (1951). (Spanish)

This paper is concerned with the study of plane figures, denoted by K^2, which are the union of two convex figures. In the first part of the paper a generalization of Helly's theorem is given: If a family of figures K^2 has the property that every 17 of the figures have a point in common, then the family has a non-empty intersection. Easy examples show this theorem to be false, however, as well as its higher dimensional generalizations. Secondly, the author solves three extremal problems. Let d and D be the minimum and maximum distances between the components of K^2, let δ_1 and δ_2 be their respective diameters and let F be the area of K^2 and L the length of its boundary. The author finds: (1) the figure maximizing F, given d and D; (2) the figure maximizing L, given d and D; (3) the circle of smallest radius containing any figure with given δ_1, δ_2, and d.
D. Gale (Copenhagen).

Santaló, L. A. Correction to the article "On pairs of convex figures". Gaz. Mat., Lisboa 14, no. 54, 6 (1953). (Spanish)

The author points out the error in the paper reviewed above which is mentioned in the review. He also gives a counter-example due to Motzkin showing that a theorem of the type sought is not possible.

Santaló, L. A. Some mean values on the hemisphere. Math. Notae 12-13, 32–37 (1952). (Spanish)

Formulae of integral geometry are used to evaluate some simple mean values on a hemisphere E of unit radius, in particular the mean distance $4/\pi$ of two points of E, the mean area $(12/\pi)-\pi$, and the mean perimeter $12/\pi$, of a triangle whose vertices lie on E. *L. C. Young.*

Grotemeyer, Karl-Peter. Die Integralsätze der affinen Flächentheorie. Arch. Math. 3, 38–43 (1952).

W. Scherrer [Comment. Math. Helv. 19, 105–114 (1946); diese Rev. 8, 339] hat Integralformeln aus der Theorie der konvexen Körper auf beliebige Flächenstücke übertragen. Der Verfasser beweist analoge Formeln für die affine Flächentheorie. Flächenintegrale, in denen u.a. die mittlere Affinkrümmung, das Affinkrümmungsmass und die Affin-

stützfunktion auftreten, werden in Randintegrale verwandelt. Als Anwendungen werden Sätze der affinen Flächentheorie im grossen, insbesondere über Flächen konstanter mittlerer Affinkrümmung bewiesen. Der von Blaschke herrührende Satz, dass Eiflächen mit dieser Eigenschaft Affinsphären sind, ergibt sich sehr einfach. *W. Fenchel.*

Grotemeyer, Karl-Peter. Eine kennzeichnende Eigenschaft der Affinsphären. Arch. Math. **3**, 307–310 (1952).

Süss, W. Über Kennzeichnungen der Kugeln und Affinsphären durch Herrn K.-P. Grotemeyer. Arch. Math. **3**, 311–313 (1952).

Mit Hilfe der Integralformeln, die in der vorstehend referierten Arbeit bewiesen sind, gewinnt Grotemeyer die folgenden Charakterisierungen der Affinsphären: Es seien ρ, K und H die Affinstützfunktion, das Affinkrümmungsmass bzw. die mittlere Affinkrümmung einer geschlossenen Fläche. Gilt dann entweder $K=1/\rho^2$ oder, die Fläche konvex vorausgesetzt, $H=1/\rho$, so ist die Fläche eine Affinsphäre. Ersetzt man ρ, K, H durch die entsprechenden Funktionen der euklidischen Differentialgeometrie, so ergeben sich Kennzeichnungen der Kugeln. Süss bemerkt, dass sich beide Sätze einem Satz der relativen Differentialgeometrie unterordnen lassen, der, auf den $(n+1)$-dimensionalen Raum verallgemeinert, folgendes besagt: Es bezeichnen $p_0=1$, p_k, $k=1, \cdots, n$, die normierten, d.h., durch die Gliederzahlen dividierten elementarsymmetrischen Funktionen der Relativkrümmungsradien einer Eihyperfläche in bezug auf eine andere. Ferner sei E die Relativstützfunktion. Gilt dann eine Relation der Form $p_{k+l}=p_lE^k$, so sind die beiden Eihyperflächen homothetisch. Der kurze Beweis beruht auf den Ungleichungen zwischen elementarsymmetrischen Funktionen positiver Zahlen und verläuft analog einem vom Verfasser [Tôhoku Math. J. **31**, 202–209 (1929)] gegebenen Beweis dafür, dass die Relativsphären durch die Konstanz einer der Funktionen p_k, $k>0$, gekennzeichnet sind. *W. Fenchel* (Kopenhagen).

Algebraic Geometry

Godeaux, L. Sur un théorème de Bertini et Laguerre concernant les quartiques gauches rationnelles. Mathesis **62**, 5–8 (1953).

Caputo, Michele. Sulla configurazione delle curve algebriche sghembe dei primi ordini dotate di $D\geqq 0$ punti doppi situate sopra quadriche. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 111–125 (1952).

The author classifies skew algebraic curves of orders 4, 5, and 6, lying on quadrics, and having only double points as singularities. By using relations between the integers representing the order, the total number of double points, the total number of circuits, etc., a table of all possible topologically distinct types is constructed. It is indicated that all of these exist and constructions are made in the more complicated cases. *G. B. Huff* (Athens, Ga.).

Segre, Beniamino. Una proprietà caratteristica in grande delle curve giacenti su di una quadrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 374–378 (1952).

The following theorem is proved: Given six linear branches in 3-space, and a plane α through their centers but not tangent to any of them, if every plane π close to α intersects the branches in six points of an irreducible conic, then the six branches [at least in neighborhoods of their centers] lie on a quadric. Corollary: The curves of order $\geqq 6$ on a quadric are the only twisted algebraic curves with ∞^3 6-secant conics. That this theorem cannot be generalized from conics to plane cubics is shown by an example of a twisted curve of order ten, not lying on a cubic surface, cut by a generic plane in ten points of a cubic. *R. J. Walker* (Ithaca, N. Y.).

***Nagell, Trygve. Problems in the theory of exceptional points on plane cubics of genus one.** Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 71–76. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

Let $Y^2=X^3-AX-B$ be the equation of an elliptic cubic C, where A and B are in a given field Ω. Then it is well known that the elliptic arguments of the points of C having coordinates in Ω constitute an Abelian group G, and that the points of C corresponding to the elements of a finite subgroup $\bar{G}$ of G are exceptional points of C over Ω. In a previous paper [Nova Acta Soc. Sci. Upsaliensis (4) **14**, no. 1 (1946); no. 3 (1947); these Rev. **9**, 100], the author has given necessary and sufficient conditions on A, B in order that C admit a group $\bar{G}$ of order $n=3, 4, 5, 6, 7$; now he studies the case $n=9$. Then, if G is non-cyclic, all the nine points of inflection of C have coordinates in Ω, the number $\sqrt{-3}$ must belong to Ω, and A, B can be expressed in the form

$$A=\sqrt{-3}cd(c^2-d^2-\sqrt{-3}cd),$$
$$4B=(c^2+d^2)(c^4+d^4-6c^2d^2-2\sqrt{-3}c^3d+2\sqrt{-3}cd^3),$$

where c, d are elements of Ω such that C has a non-zero discriminant; and vice versa. A similar result is reached, for an arbitrary Ω, in the case when C admits a cyclic group $\bar{G}$ of order 9, and gives A and B as forms in two indeterminates, with integer coefficients, of degrees 12 and 18 respectively. *B. Segre* (Rome).

Gherardelli, Francesco. Osservazioni sul gruppo dei punti $(k+1)$-pli di una g_n^k sopra una curva algebrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 398–401 (1952).

The sets of $(k+1)$-fold points of the g_n^k's contained in a g_n^r are equivalent, but for $r>k+1$ do not in general form a linear series. In an earlier paper [same Rend. (6) **6**, 286–287 (1927)] the author found the exceptional cases when $k=1$. Here he considers the situation when $k>1$, and obtains the following result: If g_n^r is simple, the sets of $(k+1)$-fold points constitute a series only when the transform of the curve by the g_n^r is a birational normal curve. *R. J. Walker* (Ithaca, N. Y.).

Manara, Carlo Felice. Sulle trasformazioni puntuali di un piano in un altro nell'intorno di un punto semplice della jacobiana. Atti Sem. Mat. Fis. Univ. Modena **5**, 40–53 (1951).

The present paper deals with a point-transformation T between two projective planes π, π' in the neighborhood of a point $0\in\pi$ belonging to the Jacobian curve of the transformation in π. If $0'\in\pi'$ corresponds to 0, the following theorems are proved: 1) A necessary and sufficient condition that T can be approximated up to and including the order n of 0 by a Cremona transformation is the existence of a linear element E_n through 0 which is transformed in $0'$;

or 2) that the straight lines through $0'$ are transformed by T^{-1} into the same element E_n.

The product of a regular transformation at 0 by a Cremona transformation not regular at 0 can be approximated by a Cremona transformation up to any order: the converse is also true. If, for a given T, n is finite, the monodromy group of T^{-1} in the neighborhood of $0'$ is the total group. In the same case a rational transformation $(1, n+1)$ exists approximating T up to and including every order $n+m$, $m \geqq 1$.

E. Bompiani (Rome).

Châtelet, François. Sur un exemple de M. B. Segre. C. R. Acad. Sci. Paris **236**, 268–269 (1953).

B. Segre [Math. Notae **11**, 1–68 (1951); these Rev. **13**, 678] proved that a non-singular cubic surface S with rational coefficients is birationally equivalent, with rational coefficients, to a plane if (1) S contains a rational set of 6 mutually skew lines, and (2) S contains a rational point. An example is given here to show that (2) is not redundant. Also, two slight generalizations of Segre's theorem are given.

R. J. Walker (Ithaca, N. Y.).

★Andreotti, A. Les problèmes de classification dans la théorie des surfaces algébriques irrégulières. Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 111–118. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

The paper is an account of the recent work of the author on irregular algebraic surfaces. The main idea employed by him is the use of a representation of the given surface into its Picard variety. [For more details see Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 380–385 (1951); these Rev. **13**, 866].

P. Abellanas (Madrid).

Baldassarri, Mario. Una condizione per l'esistenza di unisecanti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 390–397 (1952).

Ce mémoire est la suite d'un travail du même auteur sur un sujet connexe, un peu plus général [Rend. Sem. Mat. Univ. Padova **19**, 1–43 (1950); ces Rev. **13**, 578]. Avec les notations de cette précédente analyse, l'auteur approfondit ici l'étude du cas $\rho=1$.

Soit V_{r+1} une variété lieu d'un système algébrique à un paramètre de W_r^n. Si W_r^n admet une variété Δ_{r-1} multiple d'ordre s, intersection complète de W_r^n avec une forme, l'inégalité obtenue dans le précédent mémoire pour exprimer l'existence de courbes unisécantes aux W_r^n peut être améliorée en la suivante: $r+2-n/s>0$. Cette condition, en tout cas suffisante, est aussi nécessaire lorsque W_r^n est "générale pour Δ". La signification de cette expression est analogue à celle des "variétés générales" du précédent mémoire: W_r^n est "générale pour Δ "lorsqu'un certain idéal associé à W_r^n et à Δ [§§2 et 6 de la présente note] a la dimension minima.

L'auteur applique son résultat à la classification des V_3 contenant un faisceau de surfaces rationnelles, donnée par Enriques [Math. Ann. **49**, 1–23 (1897)] qui peut être améliorée sous la forme suivante: Une telle V_3 peut être transformée birationnellement en: (1) un S_3 simple; (2) un S_3 double dont la surface de branchement est une F^{2m} ayant (a) soit une droite multiple d'ordre $2m-4$, (b) soit un point multiple d'ordre $2m-2$, (c) soit deux points multiples proches d'ordre $2m-3$; (3) une V_3^m de S_4 ayant un plan multiple d'ordre $m-9$, avec un faisceau de F_2^9 à sections elliptiques, pourvues ou non d'une courbe double au plus.

L. Gauthier (Nancy).

Baldassarri, Mario. Le involuzioni ∞^d dello S_k e le loro proiezioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 530–536 (1952).

B. Segre a posé les bases de la théorie des involutions $I^d_{n,k}$ [Boll. Un. Mat. Ital. (3) **3**, 196–200 (1948); ces Rev. **10**, 566] et étudié divers exemples. Dans le présent mémoire, l'auteur étudie la projection d'une $I^d_{n,k}$ d'un point générique de S_k qui la contient, sur un hyperplan S_{k-1}. Il démontre que si $2<k\leqq d<2k-2$ et si $I^d_{n,k}$ n'est pas composée au moyen d'une $I^k_{n,k}$, la projection sur S_{k-1} est une involution $I^d_{n,k-1}$ sauf dans le cas où $I^d_{n,k}$ laisse invariants les S_{d-k+1} passant par un S_{d-k}, qui est alors lieu de points fondamentaux pour $I^d_{n,k}$.

L'itération de ce procédé sur une $I^d_{n,k}$ (n, k, d liés par les inégalités ci-dessus), qui ne laisse pas invariants les S_k passant par un S_{k-1}, pour $d-k<k<d-[\frac{1}{2}d]$ se projette suivant une $I^d_{n,[\frac{1}{2}d]+1}$, cette projection étant birationnelle. De telles involutions jouent dans cette théorie le rôle des formes. L'auteur termine en appliquant ce processus à la détermination d'une représentation analytique qui joue le rôle de la représentation monoïdale de Cayley et Halphen. Enfin, ceci permet à l'auteur de définir $I^d_{n,k}$ comme normale dans S_H lorsqu'il existe une $I^d_{n,H}$ qui engendre $I^d_{n,k}$ par projection, $H\leqq d$ étant le plus grand entier qui ait cette propriété.

L. Gauthier (Nancy).

Tibiletti, Cesarina. Un teorema fondamentale della geometria algebrica. Atti Sem. Mat. Fis. Univ. Modena **5**, 111–125 (1951).

Exposition of Noether's Theorem [cf. van der Waerden, Einführung in die Algebraische Geometrie, Springer, Berlin, 1939, Chap. VIII] with extensive bibliography.

D. B. Scott (London).

Fusa, Carmelo. Generalizzazione di un lemma di Chisini. Boll. Un. Mat. Ital. (3) **7**, 307–311 (1952).

A lemma of Chisini concerning the multiplicity of a point of maximum multiplicity of a homaloidal net is extended to linear systems of arbitrary genus and positive dimension as follows. Let $|C_n|$ be an irreducible linear system of algebraic plane curves of order n, genus p, dimension $r\geqq 1$, and grade d. If A_0 is a base point of maximum multiplicity h_0 such that $h_0\leqq n-2-2(p-1)/n$ and $d\geqq 2(p-1)+2d/(n-h_0)$ and the base points $A_1, \cdots, A_t$ in the neighborhoods of A_0 have multiplicities $h_1, \cdots, h_t$ such that $h_0+h_1+\cdots+h_t>n$, then there is a base point A of multiplicity h which is distinct from $A_0, A_1, \cdots, A_t$ and such that $2h>(n-h_0)$.

G. B. Huff (Athens, Ga.).

★Gröbner, W. La théorie des idéaux et la géométrie algébrique. Deuxième Colloque de Géométrie Algébrique, Liège, 1952, pp. 129–144. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

An expository article, not always clearly written, covering the following topics: Sums and intersections of algebraic varieties, projections, multiplicity of intersection, birational transformations, adjoint varieties, linear systems.

I. S. Cohen (Cambridge, Mass.).

Kodaira, Kunihiko. The theorem of Riemann-Roch for adjoint systems on 3-dimensional algebraic varieties. Ann. of Math. (2) **56**, 298–342 (1952).

The author's main purpose is to prove the Riemann-Roch theorem for adjoint systems on 3-dimensional algebraic varieties by means of the theory of harmonic integrals.

In the first part of the paper the author considers a compact Kähler variety $\mathfrak{M}$ of complex dimension 3 which is assumed to contain a surface S with only ordinary singularities (double, triple, and ordinary cuspoidal points), and he establishes a formula for the dimension of the vector space consisting of all triple differentials of $\mathfrak{M}$ which are multiples of $-S$. This formula is then applied to a 3-dimensional algebraic variety $\mathfrak{M}$ imbedded without singularities in a projective space. Let $|K|$ be the canonical system on $\mathfrak{M}$, introduce the arithmetic genus $a(\mathfrak{M})=r_3-r_2+r_1$ where $r_j=r_j(\mathfrak{M})$ is the number of independent $j-pl$ differentials of the first kind on $\mathfrak{M}$, and let $a(S)$ denote the corresponding virtual arithmetic genus of S (which is equal to $r_2(S)-r_1(S)$ when S is non-singular). If $S=S'+E$ where S' is a surface with ordinary singularities only, E a general hyperplane section of $\mathfrak{M}$, the Riemann-Roch formula has the form $\dim|K+S|=a(S)+a(\mathfrak{M})-1$.

For an arbitrary divisor D on $\mathfrak{M}$, the arithmetic genus $\mathcal{P}(D)$ introduced by Todd is given by the formula

$$\mathcal{P}(D)=\tfrac{1}{6}D^3+\tfrac{1}{4}KD^2+\tfrac{1}{12}(K^2+C)D-1$$

where K is the canonical divisor on $\mathfrak{M}$ ($-K$ represents the homology class dual to the first contravariant Chern class of $\mathfrak{M}$), C is a 2-cycle representing the homology class dual to the second Chern class of $\mathfrak{M}$, and where D^3, $K\cdot D^2$, $\cdots$ denote the topological intersection numbers $I(D,D,D)$, $I(K,D,D)$, $\cdots$. It is shown that $a(S)=\mathcal{P}(S)$, and hence the genus $a(D)$ for an arbitrary divisor D on $\mathfrak{M}$ may be defined by the formula $a(D)=\mathcal{P}(D)$. Next, given an arbitrary divisor D on $\mathfrak{M}$, then, for large h, $\dim|D+hE|=v(h;D,\mathfrak{M})$ is a polynomial in h of degree $n=\dim\mathfrak{M}$, and the classical arithmetic genera $p_a(\mathfrak{M})$, $P_a(\mathfrak{M})$ are defined by the formulas $p_a(\mathfrak{M})=(-1)^n v(0;0,\mathfrak{M})$, $P_a(\mathfrak{M})=v(0;K,\mathfrak{M})+1-(-1)^n$. It is shown that $p_a(\mathfrak{M})=P_a(\mathfrak{M})=a(\mathfrak{M})=\mathcal{P}(\mathfrak{M})$ where $\mathcal{P}(\mathfrak{M})=(K\cdot C/24)+1$ is the Todd genus of $\mathfrak{M}$; that is, the four definitions of arithmetic genus are equivalent for 3-dimensional algebraic varieties.

Next, a Riemann-Roch theorem is obtained which expresses the dimension of the adjoint system $|D|=|K+S|$ of a surface S on $\mathfrak{M}$ with ordinary singularities in terms of $a(\mathfrak{M})$, $a(D)$ and $a(D^2)=D^3+\tfrac{1}{2}K\cdot D^2+1$. In particular, if S is irreducible and if $\dim S\geqq 1$, the author obtains the following inequality of Severi:

$$\dim|D|\geqq D^3-a(D^2)+a(D)-a(\mathfrak{M})+2.$$

Let S be an irreducible surface on $\mathfrak{M}$ free from singularities, D an arbitrary divisor on $\mathfrak{M}$, and define the deficiency of the divisor D over S to be

$$\operatorname{def}(D/S)=\dim_S|D\cdot S|-\dim|D|+\dim|D-S|+1$$

where $\dim_S|D\cdot S|$ denotes the dimension of the complete linear system $|D\cdot S|$ on S. It is shown that $\operatorname{def}(S/S)\leqq r_1$ and that, if $|S|$ is sufficiently ample compared with D, then $\operatorname{def}(D/S)=0$ (lemma of Enriques-Severi). Finally, it is shown that there exist on $\mathfrak{M}$ exactly $2r_1$ independent simple Picard integrals of the 2nd kind and that the superabundance of the characteristic system on an irreducible non-singular surface S on $\mathfrak{M}$ is equal to the number r_2 if the complete linear system $|S|$ is sufficiently ample.

D. C. Spencer (Princeton, N. J.).

Differential Geometry

★**Finikov, S. P.** Pavlovich **Kurs differencial'noĭ geometrii. [A course of differential geometry.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 343 pp. 8 rubles.

This book contains an exposition of the elementary differential geometry of curves and surfaces. It opens with an introduction in which the ways are described in which curves and surfaces can be given in coordinate form, how singular points appear, and how contact can be described. Then follow 119 pages on plane and space curves, and 141 pages on surfaces. In both cases sharp distinction is made between differential properties of the first order and those of the second order; in curve theory there is also a section on third order properties, after which the Frenet formulas can be introduced. There are sections on the natural equations of curves and on envelopes, on the intrinsic geometry of surfaces and on the fundamental equations of a surface. Vector calculus is used throughout, and care is shown in dealing with such ticklish subjects as the theory of envelopes. There is a supplement which deals with the existence theorems on implicit functions, with the differentiation of vector functions of a scalar variable and with elementary curve plotting. A few pages are devoted to the history of the subject with special emphasis on Russian contributions. The author mentions the Moscow school of K. M. Peterson, to which B. K. Mlodzeevskiĭ and D. F. Egorov belonged, and another school dating back to N. I. Lobačevskiĭ, which is at present represented by the many Russian authors on the tensor calculus, pupils and collaborators of V. F. Kagan. The text is lucid and has a number of good figures; it also contains many exercises with answers. *D. J. Struik.*

Nádeník, Zbyněk. Les courbes de Bertrand dans l'espace à cinq dimensions. Čehoslovack. Mat. Ž. 2(77), 57–87 (1952). (Russian. French summary)

Verfasser löst das folgende Problem: unter welchen Bedingungen existiert im euklidischen Raum von fünf Dimensionen wenigstens ein Kurvenpaar, dessen begleitende Fünfbeine eine in Bezug auf euklidische Bewegungen invariante Figur bilden? Jedes solche Kurvenpaar heisst ein Bertrandpaar im Raum von 5 Dimensionen. Verfasser gewinnt acht Sätze über derartige Kurvenpaare, deren jeder auf mehrere Unterfälle führt. Die Resultate stehen mit ähnlichen im Zusammenhang, die E. Čech im euklidischen vierdimensionalen Raum gewonnen hat. Insbesondere werden notwendige und hinreichende Bedingungen dafür angegeben, dass eine Kurve mit den Kurven eines ein- oder zwei-parametrigen Systems Bertrandpaare bildet. Sind z.B. drei assoziierte Kurven vorgeschrieben, so ergibt sich für Bertrandpaare notwendig konstantes Verhalten—sei es für die Krümmungen der Kurven selbst, sei es für deren Verhältnisse. Verfasser gewinnt eine vollständige Übersicht über alle Bertrandpaare im euklidischen Raum von fünf Dimensionen. *M. Pinl* (Dacca).

Nitsche, Joachim, und Nitsche, Johannes. Ein Satz über die Normalen der Niveauflächen einer Potentialfunktion. J. Rational Mech. Anal. 2, 115–124 (1953).

Die Gesamtheit der Tangenten einer Fläche $\mathfrak{r}$ im dreidimensionalen euklidischen Raum bilden einen Strahlenkomplex. Im Parameterraum der u, v, w des Komplexes können u, v als Flächenparameter und w als Winkel einer Tangente mit $\mathfrak{r}_u$ gedeutet werden. Die Verfasser behandeln das Problem der Existenz von Potentialfunktionen, deren Niveauflächen als Normalenflächenschar eines gegebenen

Komplexes aufgefasst werden können. Es wird gezeigt: es gibt keine Potentialfunktion $F(x, y, z)$ derart, dass die Normalenkongruenzen ihrer Niveauflächen $F=\text{const.}$ sämtlich eine gegebene Fläche als Brennfläche besitzen. Die Beweisführung verwendet Cartansche Methoden und beginnt mit der Aufstellung der Ableitungsgleichungen für das begleitende Dreibein $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{n}$ (Flächennormale). Die Bestimmung einparametriger Scharen von Normalensystemen in einem Komplex erfordert die Lösung einer linearen partiellen Differentialgleichung. Eine andere Methode benutzt die folgenden Eigenschaften (im Kleinen): jeder Strahl charakterisiert durch einen Punkt im Parameterraum tritt nur einmal als Normale einer der Flächen der zu bestimmenden Normalenflächenschar auf; ferner ist der Durchstosspunkt dieser Strahlen mit der zugehörigen Normalenfläche ausgezeichnet, sofern in einem solchen jeweils ein Flächenelement orthogonal zum Strahl gegeben ist, welches die Normalenfläche tangiert. Das Bestehen dieser Vorraussetzungen führt die Verfasser schliesslich dazu die Normalenflächenschar durch eine Quadratur aus der Beziehung

$$(*)\qquad dF=f\omega^1 \quad (\omega^1=\mathfrak{a}d\mathfrak{y},\ \mathfrak{y}=\mathfrak{r}+h\mathfrak{n})$$

zu gewinnen. Wenn nun F zusätzlich der Potentialgleichung im Raume genügt, so ergibt sich neben der bereits zu (*) gehörigen Bedingung für den integrierenden Faktor f noch eine zweite derartige Bedingung. Diese beiden Bedingungen widersprechen jedoch einander. *M. Pinl* (Dacca).

Graue, Louis C. A necessary and sufficient condition that a curve lie on a hyperquadric. Proc. Amer. Math. Soc. 2, 607–612 (1951).

A necessary and sufficient condition is given in terms of the ordinary curvatures ρ_i of a curve in order that it lie on a hyperquadric. Let a hyperquadric be given by $a_{\alpha\beta}x^\alpha x^\beta=r$, $|a_{\alpha\beta}|=1$. A hyperquadric arc S is introduced by $dS^2=a_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ and a set of Frenet formulas for S containing n mutually conjugate unit vectors (with respect to $a_{\alpha\beta}$) and a system of hyperquadric curvatures R_i is derived. From these formulas a necessary and sufficient condition is found in terms of the hyperquadric curvatures. Then the problem is solved by means of invariant functions

$$f_i(R_1, \cdots, R_{n-1})=f_i(\rho_i, \cdots, \rho_{n-1}).$$

J. Haantjes (Leiden).

Tuganov, N. G. On the congruence of Dupin indicatrices of a surface. Doklady Akad. Nauk SSSR (N.S.) 88, 217–220 (1953). (Russian)

The Dupin indicatrices at the points M of a surface form a congruence. A curve along which these indicatrices admit an envelope is called a line of centers, the envelope itself a focal line. A focal line is tangent to the indicatrix at a focal point. At a point M the direction to a focal point is conjugate to the direction of the corresponding line of centers, the tangent at a focal line is parallel to the tangent to the line of centers, and the focal lines and the corresponding line of centers form a set of Combescure. The vector to the focal point P is given by

$$\mathbf{P}=\mathbf{M}\pm\frac{\mathbf{e}_1-\dfrac{R_2}{R_1}\mathbf{e}_2\cot\mu}{\left(\dfrac{1}{R_1}+\dfrac{R_2}{R_1^2}\cot^2\mu\right)^{1/2}},$$

where the line of centers is at angle μ to the line of curvature $\mathbf{e}_1$; R_1, R_2 are the radii of principal curvature. For the line of centers the equation is

$$(\omega^1)^2dR_2+2(R_2-R_1)\omega^1\omega^2\omega_1^2+(\omega^2)^2dR_1=0$$

where the ω^1, ω^2 refer to the lines of curvature (Cartan notation). We thus see that through every point M pass three lines of centers, each with corresponding two focal lines. On an indicatrix there are therefore six focal points. The natural equation of the lines of centers are

$$\frac{1}{T\rho}+\frac{a}{2}\frac{d}{ds}\ln\frac{K}{a}=0,\quad\text{or}\quad 2\tan\Theta\frac{ds}{T}+d\ln\frac{\rho K}{\cos^3\Theta}=0,$$

where a is the normal curvature, T the radius of geodesic torsion, ρ the radius of geodesic curvature of the line of centers, Θ the angle of its principal normal with the surface normal, $K=R_1R_2$.

A number of special cases are studied, as well as the congruence of curves conjugate to the lines of centers. On a quadric the lines of centers are the asymptotic curves and the curves $K=\text{const.}$ If one family of lines of centers is asymptotic, the surface is ruled. On developable surfaces any curve is a line of centers. A section of the paper is devoted to the loci of the focal lines and their relation to the surface. *D. J. Struik* (Cambridge, Mass.).

Kula, Muzaffer. Extension de la notion d'enveloppe à la géométrie réglée. Rev. Fac. Sci. Univ. Istanbul (A) 17, 322–343 (1952). (Turkish summary)

L'auteur généralise la théorie des enveloppes de familles de courbes de la géométrie différentielle ordinaire et l'étend à la géométrie réglée. L'instrument de calcul est constitué par les coordonnées dualistiques de Clifford qui s'adaptent particulièrement à la recherche. Il commence par établir la correspondance qui existe entre les courbes de la géométrie différentielle classique et les congruences synectiques de la géométrie réglée, ces dernières étant celles que l'on déduit d'une surface réglée en substituant au paramètre réel qui fixe ses génératrices un paramètre dualistique. Il cherche ensuite les conditions de raccordement d'ordre supérieur de deux congruences synectiques et d'une congruence synectique et d'une surface réglée, et déduit de ces conditions un procédé de détermination de la recticongruence tangente et de la congruence à pente dualistique constante (dont les génératrices forment un angle dualistique constant avec une droite fixe: axe) osculatrice à une surface réglée, ainsi que de remarques géométriques concernant, soit les conoïdes droits tangents à une surface réglée le long d'une génératrice, soit l'axe des surfaces à pente dualistique constante osculatrices à une surface réglée le long d'une génératrice. Il définit l'enveloppe à un paramètre (dualistique ou réel) d'une famille de congruences synectiques comme lieu des droites communes à une congruence et à la congruence infiniment voisine, et montre que, pour les recticongruences normales d'une congruence synectique cette enveloppe est la développée de la congruence. Le travail se termine par une application de la notion d'enveloppe d'une famille de congruences synectiques à la cinématique, et une généralisation du problème d'équerre de la géométrie plane (ou sphérique) relatif au glissement d'un plan (ou d'une sphère) sur un plan (ou une sphère) et de la construction de Savary relative aux enveloppes de courbes planes (ou sphèriques). *P. Vincensini* (Marseille).

Takeda, Kusuo. Principal ruled surfaces of a rectilinear congruence. J. Math. Soc. Japan 4, 286–295 (1952).

En utilisant les coordonnées plückériennes de la droite de l'espace projectif réglé R_3 et sa représentation sur la

quadrique Q_4 de R_5, et après un choix convenable du trièdre de référence, l'auteur esquisse une étude des surfaces réglées de l'espace projectif R_3 basée sur la considération de ses surfaces réglées principales.

Après avoir introduit la notion de congruence normale de l'espace projectif, et montré qu'une congruence est normale si elle n'est pas W, il envisage, en excluant les congruence non normales, les complexes quadratiques C_2 ayant un contact du 4ème ordre avec une congruence rectiligne K le long de l'un, p, de ses rayons, et introduit les surface réglées principales comme celles (au nombre de cinq) ayant le long de p un contact du 5ème ordre avec C_2, puis montre que les images des surfaces réglées principales sur Q_4 coïncident avec les lignes principales de l'image V de la congruence. Il introduit ensuite deux types importants de congruences rectilignes: les congruences s et les congruence k. Les congruences s sont définies par la propriété que parmi les cinq surfaces réglées principales issues d'un même rayon, deux divisent harmoniquement les deux développables issues du même rayon, les trois autres étant apolaires à ce rayon. Ces congruences admettent comme propriété caractéristique de donner lieu à des suites de Laplace de période 4. Les congruences k sont les congruences telles que les direction de deux de leurs surfaces réglées principales le long d'un rayon quelconque p soient celles des deux développables issues de p. L'auteur caractérise ces congruences au moyen d'une propriété des complexes linéaires C_1 osculateurs aux surfaces développables issues de p, et termine en rattachant à la considération des surfaces réglées principales les surfaces réglées quasi-asymptotiques, qu'il définit comme ayant pour images dans R_5 les quasi-asymptotiques de E. Bompiani.

P. Vincensini (Marseille).

Rozet, O. Sur certaines congruences de droites. Bull. Soc. Roy. Sci. Liège **21**, 320–327 (1952).

Ce travail reprend et complète une publication antérieure [Rozet, Acad. Roy. Belgique. Bull. Cl. Sci. (5) **19**, 179–186 (1933)]. (x) étant une surface non reglée de l'espace projectif S_3, et g une droite issue de x, non située dans le plan tangent à (x) en x et variant avec x suivant une loi déterminée, l'auteur étudie la congruence (g) des droites g; il en détermine les foyers et les plans focaux, cherche les conditions pour que (g) soit à nappes focales confondues, pour qu'elle soit W, pour que g soit une droite canonique non tangente à (x) ou que l'une des deux nappes focales de (g) se réduise à une courbe. Introduisant ensuite la congruence $(\mathfrak{g})$ formée par les conjuguées $\mathfrak{g}$ des droites g par rapport à la quadrique de Lie attachée au point x de la surface (x), il reprend pour $(\mathfrak{g})$ les questions étudiées pour (g), en faisant à cet égard tous les rapprochements que le sujet comporte.

P. Vincensini (Marseille).

Mishra, R. S. A set of $(m-n)$ congruences of curves through points of a subspace V_n of a Riemannian V_m. Ganita **3**, 95–102 (1952).

L'auteur considère, dans l'espace Riemannien V_m, un sous-espace V_n et un système de $(m-n)$ congruences de courbes telles que, par chaque point de V_n, il passe une courbe et une seule de chaque congruence. Il recherche les expressions de la tendance et de la divergence d'un vecteur le long d'une courbe de l'une quelconque des $(m-n)$ congruences. Il en déduit quelques propriétés se rattachant à ces deux notions, ainsi que quelques résultats relatifs aux lignes de courbure généralisées, et aux directions conjuguées ou asymptotiques généralisées d'une sous-espace V_n de V_m.

P. Vincensini (Marseille).

Jonas, Hans. Zur Theorie der W-Kongruenzen mit ebener Mittelfläche. Math. Nachr. **9**, 1–21 (1953).

Diese Arbeit handelt über W-Kongruenzen für welche die Ortsfläche des Strahlmittelpunktes eine Ebene ist. Zuerst wird die bekannte Tatsache bewiesen, dass man zur Konstruktion dieser Kongruenzen eine nicht definite quadratische Differentialform $a^2du^2-b^2dv^2$ mit der Krümmung Null zu bestimmen hat, wobei a und b überdies durch eine Relation verbunden sind. Sodann wird ein neuer Weg zur Konstruktion dieser Kongruenzen angegeben als auch der Zusammenhang zwischen den beiden Lösungen des Problems. Als Sonderfälle betrachtet Verf. die W-Normalenkongruenzen mit Mittelebene und die unter dem Problem fallenden W-Kongruenzen, die auch Darboux-Kongruenzen sind. Im letzten Falle lassen die Brennflächen unendlich viele R-Netze zu. Es zeigt sich, dass unter diesen Brennflächen sich gewisse Minimalflächen finden, insbesondere die Ennepersche. Aus den erhaltenen Resultate werden ausserhalb der ursprünglichen Fragestellung noch einige allgemeine Eigenschaften der Enneperschen Minimalfläche abgeleitet.

L. Haantjes (Leiden).

Löbell, Frank. Zusammenhänge zwischen den Theorien der Kurvenkongruenzen und der Flächenabbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 47–50 (1953).

Consider a congruence of oriented curves in Euclidean 3-space, and let e denote the field of unit tangent vectors. At any point P consider a surface x perpendicular to x and P. Let x be mapped on the unit sphere by making correspond to any point x the point of the unit sphere whose position vector is the vector e for the point x. The paper interprets the fundamental quantities of this surface mapping [e.g., Löbell, same S.-B. **1947**, 15–23 (1949); these Rev. **11**, 130] in terms of the fundamental quantities of the congruence [Löbell, Math. Z. **56**, 208–218 (1952); these Rev. **14**, 315].

S. B. Jackson (College Park, Md.).

Löbell, Frank. Flächen mit vorgegebener vektorieller Differentialinvariante. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1952**, 99–101 (1953).

If $x=x(u,v)$ is a surface in Euclidean 3-space, the vector expression $x_u \times x_v$ is a differential invariant of weight one. The author here discusses the problem of determining all surfaces for which this invariant is a prescribed vector function of u and v. It is pointed out that the problem is in some ways analogous to, but is actually distinct from, that of determining all surfaces with a given first fundamental form. Actually, since the invariant determines both the curvature K and the spherical image, the problem is more closely allied to that of determining the surfaces for which the third fundamental form and the curvature are prescribed.

S. B. Jackson (College Park, Md.).

Barner, Martin. Zur projektiven Differentialgeometrie der konjugierten Netze im vierdimensionalen Raum. Arch. Math. **3**, 409–420 (1952).

Let $\mathfrak{p}_i(u,v)$ $(i=0,1,\cdots,4)$ be the vertices of a pyramid for each pair of the parameters u, v in a four-dimensional projective space R_4. A scalar or form ω in $\mathfrak{p}$'s is called a half-invariant form of weight $(c_0,c_1,\cdots,c_4)$ if under the transformation $\mathfrak{p}_i^*=\rho_i(u,v)\mathfrak{p}_i(u,v)$ $(i=0,1,\cdots,4)$ it is transformed into $\omega^*=\rho_0^{c_0}\cdots\rho_4^{c_4}\omega$. Let π_i be Pfaffian forms in $\mathfrak{p}$'s with $\pi_i^*=\pi_i-d(\log\rho_i)$ $(i=0,1,\cdots,4)$; then the half-invariant exterior differential of ω with respect to π_i is defined by $\check{d}\omega=d\omega+\sum c_i\pi_i\wedge\omega$, where d denotes the ex-

terior differentiation of Cartan and $\wedge$ the alternating multiplication, so that $\check{d}(\check{d}\omega)=\sum c_i d\pi_i\wedge\omega$. Choose the point $d\mathfrak{p}_i+\pi_i\mathfrak{p}_i$ to be on the line of intersection of the tangent plane at $\mathfrak{p}_i$ and the face of the pyramid opposite to $\mathfrak{p}_i$, so that

$$(*)\qquad \check{d}\mathfrak{p}_i=\sum_{k=0}^{4}\mathfrak{p}_k\omega_{ik},\quad \omega_{ii}=0\quad (i=0,1,\cdots,4),$$

with the integrability conditions

$$(**)\qquad d\pi_i=-\sum_{k\neq i}\omega_{ik}\wedge\omega_{ki},\quad \check{d}\omega_{il}=\sum_{k\neq i,l}\omega_{ik}\wedge\omega_{kl}.$$

If the surface described by $\mathfrak{p}_0$ sustains a conjugate net $N_{\mathfrak{p}_0}$, then the fundamental equations (*) and the integrability conditions (**) can be simplified by taking $\omega_{01}=0$, $\omega_{02}=0$ to be the two curves, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ to be the Laplace transformed points, and $\mathfrak{p}_3$ and $\mathfrak{p}_4$ to be respectively in the osculating planes of the two curves of $N_{\mathfrak{p}_0}$ at $\mathfrak{p}_0$. The author uses this pyramid of reference to study the differential projective geometry of conjugate nets in the space R_4, and obtains the conditions for this pyramid to be those introduced by the reviewer and other authors. *C. C. Hsiung.*

Leichtweiss, Kurt. Natürliche Gleichungen einer Fläche. Math. Z. 57, 244–264 (1953).

Consider an analytic surface, S, a point P on S, and an analytic curve C on S passing through P. Let there be given an analytic quadratic differential form, Ω, in a neighborhood of P. Then there exists a unique parameter system (p, q) in this neighborhood such that: (1) the parameter system is analytic; (2) $\Omega=D(p,q)(dp^2+\xi dq^2)$ where ξ is the sign of the discriminant of Ω at P; and (3) along C we have $q=0$ and p equal to the arc length of C. When Ω is the first, second, or third fundamental form of S, these parameters are called normed isothermal, affine-isothermal, or spherical-isothermal, respectively.

It is then proved that with certain exceptions there exists locally a unique analytic surface which (1) passes through a given analytic strip Γ, (2) has (p, q) as isothermal, affine-isothermal, or spherical-isothermal parameters, respectively, and (3) has a given linear fractional function of K and H equal to a given analytic function $f(p, q)$. This leads to a generalization of a theorem of Scherrer concerning relative differential forms [Süss, Arch. Math. 2, 103–104 (1950); these Rev. 11, 616], and to a new proof of a theorem of Darboux concerning the realization of an analytic, two-dimensional Riemann space as a surface.

C. B. Allendoerfer (Seattle, Wash.).

Minagawa, T., and Rado, T. On the infinitesimal rigidity of surfaces. Osaka Math. J. 4, 241–285 (1952).

The paper considers surfaces $\mathfrak{x}$ of class C'' in E^3 with $EG-F^2>0$, represented vectorially in the form $\mathfrak{x}(u,v)$. Modify $\mathfrak{x}$ by means of a deformation vector $\mathfrak{z}(u,v)$, i.e., form $\mathfrak{x}(u,v)+\epsilon\mathfrak{z}(u,v)$. If l_ϵ is the length of the curve $u(t)$, $v(t)$ on $\mathfrak{x}+\epsilon\mathfrak{z}$ and $dl_\epsilon/d\epsilon$ vanishes for $\epsilon=0$ for every curve $u(t)$, $v(t)$, then $\mathfrak{z}$ defines an infinitesimal (isometric) deformation of the surface $\mathfrak{x}$. If $\mathfrak{x}$ possesses no infinitesimal deformation other than the trivial $\mathfrak{z}=(\mathfrak{a}\times\mathfrak{x})+\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{b}$ constant, then $\mathfrak{x}$ is infinitesimally rigid.

Closed convex surfaces on which the points with positive Gauss curvature are dense are proved to be infinitesimally rigid. This was known previously only under the assumptions that both $\mathfrak{x}$ and $\mathfrak{z}$ are of class C'''. If a finite number of holes are cut into a closed convex surface, then the remainder of the surface is infinitesimally rigid under deformations $\mathfrak{z}$ which leave the boundary pointwise fixed ($\mathfrak{z}=0$ on the boundary). If the piece inside each hole lies in a tangent plane of the surface, then it is not necessary to require that the boundary be kept fixed, provided the surface is of class C''' in a neighborhood of the boundary. This is an improvement of well-known results of Rembs. A finitely connected piece of a quadratic surface is infinitesimally rigid when any subarc of the boundary is kept pointwise fixed.

The paper then passes to surfaces of negative curvature, where no results on infinitesimal rigidity had been known. A surface $\mathfrak{x}$ of negative curvature, where the parameter region is a compact domain in the (u, v)-plane bounded by a finite number of closed Jordan curves (with certain restrictions) and u, v are asymptotic parameters, is infinitesimally rigid under deformations $\mathfrak{z}$ vanishing on the boundary. If the parameter region is doubly connected and the outer curve is strictly convex and the inner curve is a rectangle (on the surface this means a quadrangle whose sides are asymptotic lines), then the surface is infinitesimally rigid under $\mathfrak{z}$ which vanish on the outer curve.

Certain rigidity theorems are proved for surfaces with vanishing curvature and finally the torus is proved to be infinitesimally rigid. The last result had been proved by Liebmann for analytic $\mathfrak{z}$. *H. Busemann.*

Yanenko, N. N. Certain necessary criteria for deformable surfaces V_m in an $(m+q)$-dimensional Euclidean space. Trudy Sem. Vektor. Tenzor. Analizu 9, 236–287 (1952). (Russian)

The paper mainly supplies detailed proofs for results announced by the author [Doklady Akad. Nauk SSSR (N.S.) 72, 857–859, 1025–1028 (1950); 82, 685–688 (1952); these Rev. 12, 357; 14, 87]. Noteworthy among the additional results is the observation that the author's work contains Allendoerfer's theorem [Amer. J. Math. 61, 633–644 (1939); these Rev. 1, 28] that the type of a nonrigid manifold cannot exceed two. *H. Busemann.*

✶Salenius, Tauno. Über dreidimensionale geschlossene Räume konstanter negativer Krümmung. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 107–112. Johan Grundt Tanums Forlag, Oslo, 1952. 27.50 kr.

The author shows how to obtain closed 3-dimensional Riemannian manifolds of constant negative curvature by identifying pairs of faces of a polyhedron. Infinitely many such 3-manifolds are so constructed, the simplest of which is based on a polyhedron with 68 faces. *S. B. Myers.*

Lopšic, A. M. An algebraic problem of the theory of Riemannian spaces of the first class. Trudy Sem. Vektor. Tenzor. Analizu 9, 462–490 (1952). (Russian)

The paper is concerned with the necessary and sufficient algebraic conditions that the tensor K_{ijkl} of a V_n must satisfy in order to be the curvature tensor of a hypersurface of a euclidean E_{n+1}. The problem is one of existence of a symmetric solution for α of $K_{ijkl}=\alpha_{ik}\alpha_{jl}-\alpha_{il}\alpha_{jk}$. The results are not new, but are more completely stated in terms of multilinear forms whose coefficients are the components of the curvature tensor and its various contractions.

M. S. Knebelman (Pullman, Wash).

Tsuboko, Matsuji. On a two-dimensional space of a projective connection associated with a surface in R_3. Osaka Math. J. 4, 101–112 (1952).

Let R_n be an n-dimensional space with a projective connection. The development of a curve in a space R_2 is studied

by a method analogous to that used in establishing the ordinary projective theory of a plane curve. A space R_2 is introduced to a surface in a space R_3 by a projection, and there are obtained some properties of the space R_2 and some relations between these two spaces R_2 and R_3.

C. C. Hsiung (Bethlehem, Pa.).

Verbickiĭ, L. L. Geometry of conformal Euclidean spaces of class 1. Trudy Sem. Vektor. Tenzor. Analizu **9**, 146–182 (1952). (Russian)

This paper is concerned with hypersurfaces C_n^1 of Euclidean space E_{n+1} whose Riemann metric is positive definite and which are conformally flat ($n \geqq 4$). The main theorem is that all but possibly one of the radii of principal curvatures are necessarily equal; that is, all points of C_n are pseudoumbilical. The proof is based on the equations of Gauss and Codazzi; the second fundamental form of C_n^1 being $\pi_{ij} = \sigma g_{ij} + (\bar{\sigma} - \sigma) e_i e_j$, where σ, $\bar{\sigma}$ are the principal curvatures and e_i are unit vectors (σ is any one of the $n-1$ equal curvatures; if $\bar{\sigma} = \sigma$ the space is of constant curvature). Another result obtained is that every conformally flat hypersurface is the envelope of a one-parameter family of hyperspheres and if such a C_n^1 is to be subprojective the centers of these spheres must be collinear.

M. S. Knebelman (Pullman, Wash.).

Šapiro, Ya. L. Spaces containing projective systems of curves. Trudy Sem. Vektor. Tenzor. Analizu **6**, 494–505 (1948). (Russian)

The essential problem treated in the present paper is imbedding of a given system of curves in an n-dimensional manifold X_n into an affinely connected space A_{n+m} of dimensions $n+m$. In the first part of the paper we find at first the relations between the concept of "geodesic field of directions" introduced by the present author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 237–239 (1941); these Rev. **3**, 191] and projective differential geometry. A field of directions defined by v^α ($\alpha = 0, 1, \cdots, n$) is called geodesic when the family of its trajectories (v-lines) passing through all points of any geodesic of A_{n+1} composes entirely a geodesic surface. The relations $v^\alpha{}_{,\beta} = T\delta_\beta{}^\alpha + B_\beta v^\alpha$ and $v^\lambda R^\alpha{}_{(\sigma\beta)\lambda} = Q_{(\sigma}\delta_{\beta}{}^{\alpha}{}_{)} + S_{\sigma\beta} v^\alpha$ are necessary and sufficient for the geodesic field of directions v^α, where $v^\alpha{}_{,\beta}$ is the covariant derivative of v^α and T, B, Q, S are certain quantities in A_{n+1} with the curvature tensor $R^\alpha{}_{\sigma\beta\lambda}$. When A_{n+1} admits a geodesic field of directions v^α, we can choose such a coordinate system that its x^0-lines coincide with v-lines and the system of geodesics in A_{n+1} is induced to the system of projective geodesics in the subspace X_n defined by $x^0 = \text{const}$. If the equations of a given system of paths in X_n are contained in the equations of geodesics in A_{n+1}, the system is said to be imbedded in A_{n+1}, which admits necessarily a geodesic field of directions. These facts lead us to the result: A given system of paths in X_n can be imbedded in A_{n+1}, when and only when the system is a projective system of geodesics. We find also the relations of imbedding A_{n+1} to the projective coordinate system of van Dantzig and E. Cartan. The theory may be generalized to A_{n+m}. In the second part it is shown that there are three types of systems of curves defined by $d^2x^i/dt^2 = f^i(x, dx/dt)$ which can be imbedded in A_{n+1}, where $i = 1, 2, \cdots, n$. Without calculation, coefficients of connection of the imbedding A_{n+1} are stated for each of the three types. Finally, there are some statements concerning an A_{n+1} which admits at the same time a one-parameter group of affine collineations and a projective system of trajectories.

A. Kawaguchi.

Norden, A. P. On polar normalization in a space with a degenerate absolute. Trudy Sem. Vektor. Tenzor. Analizu **9**, 198–212 (1952). (Russian).

This is a continuation of the author's investigations of the past half a dozen years in the geometry of normalized spaces. An n-dimensional projective space with a hypersurface Q_{n-1} of the second order as absolute is normalized if with each point of a manifold X_k one associates two linear spaces P_{n-k} and P_{k-1}, the first having only this point in common with the tangent space P_k while the second is in P_k but does not contain the point. In order that this normalization be polar, P_{n-k} and P_{k-1} must be polar with respect to Q_{n-1}.

In the present paper one applies the principle of duality to the above using a hypersurface Q_{n-1} of the second class and the linear spaces P_{n-k} and P_{k-1} are replaced by pencils of hyperplanes Π_{n-k} and Π_{k-1}. If $\xi = \xi(u^1, \cdots, u^k)$ defines X_k and $\xi, \xi_1, \cdots, \xi_{n-k-1}$ together with ξ define Π_{n-k}, while $\eta_i = \partial_i \xi - l_i \xi$ define Π_{k-1}, then the $n+1$ hyperplanes are independent. If the absolute Q_{n-1} is defined by $a^{\alpha\beta}\eta_\alpha\eta_\beta = 0$ and $\eta\zeta = a^{\alpha\beta}\eta_\alpha\zeta_\beta$ then $\xi\eta_i = \bar{\xi}\eta_i = \xi_s\eta_i = 0$. The further normalization of the vectors depends on whether X_k belongs to Q_{n-1} or does not (in the first case $\bar{\xi}\xi = \pm 1$, in the second, $= 0$). In either case the construction of a metric depends on the order of degeneracy of the absolute, r, which is the number of independent hyperplanes satisfying $a^{\alpha\beta}\zeta_\beta = 0$. The intrinsic geometry of X_k in this case admits a metric g_{ij} such that $\nabla_k g_{ij} = 2l_k g_{ij}$ and r independent vectors g^i such that $\nabla_k g^i = l_k g^i$, $g_{ij}g^j = 0$. The non-degenerate case has non-zero curvature, while if $r = 1$ the curvature is zero. The main result in this case is that the intrinsic geometry of X_k is that of Laguerre.

M. S. Knebelman (Pullman, Wash.).

Yaglom, I. M. On linear subspaces of symplectic space. Trudy Sem. Vektor. Tenzor. Analizu **9**, 309–318 (1952). (Russian)

By a symplectic space we mean a real linear space of even dimensions $2n$ in which scalar product of vectors is defined by a non-degenerate skew-symmetric bilinear form $(x, y) = g_{\alpha\beta}x^\alpha y^\beta$ ($\alpha, \beta = 1, 2, \cdots, 2n$), where $g_{\beta\alpha} = -g_{\alpha\beta}$, i.e., $(y, x) = -(x, y)$. In this space the symplectic group [see, e.g., H. Weyl, The classical groups, Princeton, 1939; these Rev. **1**, 42] plays the same role as the group of motions in an euclidean space. A linear subspace U of a symplectic space is characterized by two values, i.e., its dimension and its defect (=dimension number of the isotropic subspace contained in it). It is called zero subspace if $p = s$, or symplectic if $s = 0$. Then discussion is made for classification and volume of a polyvector in a subspace and for the degree of parallelism as well as orthogonalism of two linear subspaces. There are found many kinds of scalar products of polyvectors in the symplectic space and of angles between two polyvectors. Finally, making use of these results, the system of invariants of two linear subspaces are obtained, e.g., symplectic angles, stationary 2-directions, etc., analogously as in the euclidean space.

A. Kawaguchi (Sapporo).

Raševskiĭ, P. K. On the geometry of homogeneous spaces. Trudy Sem. Vektor. Tenzor. Analizu **9**, 49–74 (1952). (Russian)

The purpose of the paper is to implement Klein's program for geometry. A real Lie group G_r being given, the problem is to construct a homogeneous space K_n for which G_r is the fundamental group. This problem was solved by Cartan for semi-simple groups and the present author's work, though not completely successful, is a considerable forward step. It is assumed that G_r is transitive having a stationary

subgroup H_m so that $m=r-n$; then it is possible to introduce in K_n an affine connection which is invariant under G_r and is symmetric (in Cartan's sense), i.e., $\nabla_p R^i{}_{jkl}=0$, $\nabla_p S^i{}_{jk}=0$, where R is the curvature tensor and S is the torsion. This connection is expressible in terms of certain vectors and the structure constants of the group. For the study of affine homogeneous spaces it is assumed that the Cartan metric of G_r, $g_{ij}=C^\alpha{}_{i\beta}C^\beta{}_{j\alpha}$ is non-degenerate on H_m. Then the sufficient conditions are given by: if G_r of a homogeneous space K_n is semi-simple and compact, K_n is affine homogeneous. A similar theorem holds if H_m is semi-simple. The author also proves some known theorems on metric spaces and gives some illustrations.

M. S. Knebelman (Pullman, Wash.).

Atanasyan, L. S. Rigged manifolds of a particular form in a multidimensional affine space. Trudy Sem. Vektor. Tenzor. Analizu 9, 351–410 (1952). (Russian)

In a flat affine space E_n a manifold V_m is defined by its radius vector $\mathbf{r}=\mathbf{r}(x^i)$, $i, j, k=1, \cdots, m$, all $\mathbf{r}_i=\partial\mathbf{r}/\partial x^i$ linearly independent. The rigging (Einspannung) is performed by $n-k$ vectors $\underset{\kappa}{\xi}$, $\kappa=1, \cdots, n-m$, such that the $\mathbf{r}_i$, $\underset{\kappa}{\xi}$ are linearly independent. Then $\mathbf{r}_{ij}=\Gamma^h{}_{ij}\mathbf{r}_h+\overset{\kappa}{h}_{ij}\underset{\kappa}{\xi}$, or $\mathbf{r}_{i\|j}=\overset{\kappa}{h}_{ij}\underset{\kappa}{\xi}$, where $\mathbf{r}_{i\|j}$ is the covariant derivative. Moreover, $\underset{\kappa}{\xi}_i=\underset{\kappa}{\beta}_i{}^h\mathbf{r}_h+\underset{\kappa}{\overset{\lambda}{\gamma}}_i\underset{\lambda}{\xi}$. Between the $\Gamma^h{}_{ij}$, $\overset{\kappa}{h}_{ij}$, $\underset{\kappa}{\beta}_i{}^h$, $\underset{\kappa}{\overset{\lambda}{\gamma}}_i$ exist the Gauss-Codazzi-Ricci equations. These equations are written out, as well as the transformation equations of these tensors under an arbitrary change of rigging and a change in which the $\underset{\kappa}{\xi}$ remain in the same plane Π_{n-m}. A necessary and sufficient condition that a V_m lie in an $E_{m+p}\subset E_n$ is that a rigging exist for which $\overset{\rho}{h}_{ij}=0$, $\underset{\alpha}{\overset{\rho}{\gamma}}_i=0$, $\alpha=1, \cdots, p$, $\rho=p+1, \cdots, n-m$.

A number of special cases are investigated.

a) Developable V_m of rank r, which are V_m consisting of ∞^r generating E_{m-r} and which at all points of each of these E_{m-r} have one and only one tangent E_m. A necessary and sufficient condition is that the spaces of all tensors $\overset{\kappa}{h}_{ij}$ at every point are imbedded in some plane Π_r, where r is the smallest number satisfying this condition. Along each generating E_{m-r} the osculating space (spanned by $\mathbf{r}_i$, $\mathbf{r}_{ij}$) is the same.

b) s-Fold degenerated manifolds of rank r, which are V_m consisting of ∞^r generating V_{m-r} such that at all points of a generator the tangent space to the V_m is parallel to a plane E_s $(n>s\geqq m)$. A necessary and sufficient condition is that a rigging exist for which the tensors $\overset{\rho}{h}_{ij}$, $\underset{\alpha}{\overset{\rho}{\gamma}}_i$ $(\alpha=1, \cdots, s-m, \rho=s-m+1, \cdots, n-m)$ at every point are imbedded in some r-dimensional space. The vectors $\underset{\alpha}{\xi}$ are parallel to the flat E_s corresponding to the generator passing through this point.

c) The trivial and the axial type of rigging. In the trivial case the space of the $\underset{\kappa}{\xi}$ is at all points parallel to a fixed plane Π_{n-m}. Then $\underset{\kappa}{\beta}_i{}^h=0$ for all κ and the induced Riemann tensor is zero. In the axial case all the $(n-m)$-spaces of the rigging of the V_m pass through one and only one plane Π_{k-1}. In this case $\underset{\kappa}{\beta}_i{}^h=\underset{\kappa}{\varphi}\delta_i{}^h$, $\kappa=1, \cdots, k$, where the $\underset{\kappa}{\varphi}$ are invariants. The tensor of Ricci $R^h{}_{ijh}$ is symmetrical.

Then follow a series of algebraic theorems dealing with systems of k tensors $\overset{\kappa}{h}_{ij}$ and k tensors $\underset{\kappa}{\beta}_i{}^h$, $\kappa=1, \cdots, k$, forming an exterior orthogonal system (Cartan), that is, a system for which $\overset{\kappa}{h}_{i[j}\underset{\kappa}{\beta}^h{}_{k]}=0$. They involve so-called coplanar systems $\overset{\kappa}{h}_{ij}$, for which the vectors $\overset{\kappa}{h}_{ij}x^j$ are linearly dependent for any vector x^j; when $k=2$ the tensors are collinear. These theorems are applied to another set of special manifolds. They involve manifolds admitting riggings which induce a connection of zero curvature, a projectively euclidean connection, an equiprojective connection (here the V_m is in E_{m+1} and E_{m+2}). An example is the theorem that, when for $n<2m$ and $n-m<5$ and the $\underset{\kappa}{\beta}_i{}^h$ linearly independent, a nontrivial rigging of a V_m in E_n induces a connection of zero curvature, then the V_m is developable of rank r $(r\leqq n-m)$ and along every plane generator the rigging spaces are parallel to each other. Here r is the dimension of the common region of the $\overset{\kappa}{h}_{ij}$, $\underset{\kappa}{\beta}_i{}^h$ with respect to the lower indices. For related work see the papers of A. M. Lopšic [same Trudy 8, 273–285, 286–295 (1950); these Rev. 12, 636, 637]. Also quoted is the Moscow dissertation of G. F. Laptev (1941).

D. J. Struik (Cambridge, Mass.).

Atanasyan, L. S. On some manifolds of special form imbedded in a centro-affine space. Doklady Akad. Nauk SSSR (N.S.) **88**, 189–192 (1953). (Russian)

In the paper reviewed above the author has defined s-fold degenerated V_m of rank r. The rigging of such a V_m is called central if every rigging $(n-m)$-space passes through the center of the affine space E_n. Then one of The rigging vectors can be taken as $\underset{n-m}{\xi}=\mathbf{r}$. When in this case the tensors $\overset{\nu}{h}_{ij}$ $(\nu=1, 2, \cdots, n-m-1)$ lie in some r-space (r the smallest number possible), then the V_m is an $(m+1)$-fold degenerated manifold of rank r. Every generating V_{m-r} lies in some plane E_{m-r+1} passing through the center of E_n. The tangent space to the V_m along these generating V_{m-r} is imbedded in some $E_{m+1}\supset E_{m-r+1}$ also passing through this center. An example is the V_2 in R_4 with $\overset{1}{h}_{ij}$ of rank 1, which is a 3-fold degenerated manifold of rank 1, hence built up of ∞^1 plane curves; the plane of each curve passes through the center. The tangent planes along every curve are imbedded in some three-space also passing through the center. If $\overset{1}{h}_{ij}=0$, the V_2 lies in an E_3. The paper also discusses the case in which the generating V_{m-r} are totally geodesic.

D. J. Struik (Cambridge, Mass.).

Yano, Kentaro, and Hitosi Hiramatu. On the projective geometry of K-spreads. Compositio Math. 10, 286–296 (1952).

In this paper the authors introduce some improvements into the projective geometry of K-spreads. They do this by using what they call a semi-natural frame of reference. In the geometry of K-spreads as studied by Douglas [Math. Ann. 105, 707–733 (1931)] coefficients of projective connection are used whose law of transformation is complicated, which makes it difficult to deduce tensors from it. In this paper the authors consider the problem of determining a projective connection with respect to which the system of K-spreads considered will be a system of K-dimensional flat subspaces of the kind considered by Chern [Proc. Nat. Acad. Sci. U. S. A. 29, 38–43 (1943); these Rev. 4, 259].

Appropriate curvature tensors are obtained. It is shown that the results of previous writers on this subject are obtained by imposing certain restrictive conditions upon their semi-natural frame of reference. *E. T. Davies.*

★Dolbeault, P. Formes différentielles méromorphes sur les variétés kählériennes compactes. Deuxième Colloque de Géometrie Algébrique, Liège, 1952, pp. 83–87. Georges Thone, Liège; Masson & Cie, Paris, 1952. 375 Belgian francs; 2625 French francs.

The author considers various problems of Cousin type for a compact Kähler manifold V. Let r be a finite covering of V by open sets U_j, where U_j and $U_j \cap U_k \neq 0$ are homeomorphic to open cells, and suppose given in each U_j a meromorphic differential p-form Π_j. a) Under what circumstances does there exist a meromorphic differential form Π of degree p on V such that $\Pi_j - \Pi = F_j$ is holomorphic in U_j? If Π exists, then $\Pi_j - \Pi_k = F_{jk}$ is holomorphic in $U_j \cap U_k$. If Π_j and Π are closed, Π is a form of the 3rd kind with the singular parts Π_j. b) A meromorphic form ϖ on V is of the 2nd kind if in U_j it is equal to $d\Pi_j$; then $\Pi_j - \Pi_k = \Phi_{jk}$ is meromorphic and closed. The form ϖ is said to be distinguished if the Φ_{jk} are holomorphic (distinguished data).

Problem 1: Given holomorphic p-forms F_{jk} in $U_j \cap U_k$ satisfying $F_{jk} + F_{ki} + F_{ij} = 0$ (that is, given a 1-cocycle on the nerve N_r of the covering r with coefficients in the faisceau of germs of holomorphic p-forms), to find F_j holomorphic in U_j and such that $F_j - F_k = F_{jk}$. This problem contains a). There exist forms G_j of type $(p, 0)$ harmonic in U_j such that $G_j - G_k = F_{jk}$, and the harmonic form $\mathfrak{L}^{(p,1)}$ of type $(p, 1)$ on V which is equal to $\bar{\partial} G_j$ in U_j is uniquely determined by the data. We say that F_{jk} and $\mathfrak{L}^{(p,1)}$ are associated, and Problem 1 is solvable if and only if the associated $\mathfrak{L}^{(p,1)}$ is 0, in which case the solution is determined up to a holomorphic p-form on V. Conversely, given $\mathfrak{L}^{(p,1)}$, there exist forms G_j of type $(p, 0)$ defined respectively in U_j, $dG_j = \mathfrak{L}^{(p,1)} | U_j$.

Problem 2: Given $\mathfrak{L}^{(p,1)}$, do there exist p-forms Π_j defined respectively in U_j such that $F_{jk} = \Pi_j - \Pi_k$ are holomorphic and associated with $\mathfrak{L}^{(p,1)}$? Problem 2 has a solution if: i) there exists an irreducible non-singular subvariety Y and a number α such that $\Omega \sim \alpha Y$ in the sense of currents where Ω is the fundamental Kähler 2-form; ii) $0 \leqq p \leqq \dim V - 2$.

Two forms of the 2nd kind belong to the same class if their difference is equal to $\omega + d\Pi$ where ω is holomorphic, Π meromorphic on V. Let $M^{(p,1)}$ be the vector space of the harmonic forms of type $(p, 1)$ on V, $m^{(p,1)}$ the subspace of forms associated with $F_{jk} = \Pi_j' - \Pi_k'$ where Π_j' is meromorphic and closed on U_j. The vector space of the classes of $(p+1)$-forms of the 2nd kind which are distinguished is canonically isomorphic to a subspace of $M^{(p,1)}/m^{(p,1)}$. If $p=0$, every form of the 2nd kind is distinguished and $m^{(0,1)} = \{0\}$. If V satisfies conditions i) and ii) of Problem 2, the preceding isomorphism is onto $M^{(p,1)}/m^{(p,1)}$ and the dimension of the vector space of classes of 1-forms of the 2nd kind is one-half the first Betti number of V.

If the F_{jk} and F_j are closed, there exist holomorphic $(p-1)$-forms f_{jk} on $U_j \cap U_k$ such that $df_{jk} = F_{jk}$, and the closed $(p-1)$-form $c_{jki} = f_{jk} + f_{ki} + f_{ij}$ defines a 2-cocycle $C(r)$ of N_r with coefficients in the faisceau of germs of closed holomorphic $(p-1)$-forms. The 2-cocycle $C(r)$ determines a 2-class c of the cohomology of V, and $c=0$ if the problem has a solution. Conversely, $c=0$ implies that there is a refinement r' of r on which the data induce f'_{jk} satisfying $f'_{jk} + f'_{ki} + f'_{ij} = 0$. Since V is compact Kähler, there exist G_j' of type $(p, 0)$ harmonic in U_j such that $G_j' - G_k' = f_{jk}$ and $\partial G_j'$, which is holomorphic, gives a solution of the problem for r'. *D. C. Spencer* (Princeton, N. J.).

Rozenfel'd, B. A. Spinor representations of real rotations. Trudy Sem. Vektor. Tenzor. Analizu 6, 506–514 (1948). (Russian)

R. Brauer and H. Weyl [Amer. J. Math. 57, 425–449 (1935)] found all spinor representations for the groups of complex euclidean notions for any number of dimensions. The present paper shows how to settle the analogous problem for real groups of motions in euclidean and pseudo-euclidean (euclidean with different signature) spaces. The real orthogonal group is O^n, the corresponding group for signature $n-2l$ is O_l^n. The corresponding twice covering groups are the spinor group S^n and S_l^n. Introduced is a Clifford algebra C^n of 2^{n-1} basic elements built up of $e_1, e_2, \cdots, e_{n-1}$ such that $e_i^2 = -e$, $e_ie_j = -e_je_i$. For C_l^n the corresponding formulas are $e_i^2 = -\sigma(i-l)e$, $e_ie_j = -e_je_i$, where $\sigma(n) = -1$ for $n \leqq 0$, and $+1$ for $n > 0$. An element a of C_l^n is given by

$$a = a_0e + a_ie_i + a_{ij}e_{ij} + \cdots + a_{i_1\cdots i_k}e_{i_1\cdots i_k} + \cdots + a_{12\cdots n-1}e_{12\cdots n-1},$$

where the $i_1, \cdots, i_k$ run from 1 to $n-1$, and

$$e_{i_1i_2\cdots i_k} = e_{i_1}e_{i_2}\cdots e_{i_k}.$$

One of the results is expressed in the following theorem. The algebra C_l^n for $n = 2m+1$ is isomorphic to the algebra $R^{2^{m-1}}[i, j, k]$ of 2^{m-1}-dimensional quaternionic matrices, or the algebra R^{2^m} of 2^m-dimensional real matrices. For $n = 2m$ it is isomorphic to the algebra $R^{2^{m-1}}[i]$ of 2^{m-1}-dimensional complex matrices, the algebra $2R^2$ of 2^{m-1}-dimensional dual matrices, or the algebra $2R^{2^{m-2}}[i, j, k]$ of 2^{m-2}-dimensional dual quaternionic matrices.

Dual numbers are $a + b\omega$, $\omega^2 = +1$; i is the ordinary complex unit. For instance, $C^3 = C_0^3$ is isomorphic to the $R[i, j, k]$ of quaternions, C_1^3 to the R^2 of two-dimensional real matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, for $n = 4m+1$ and l even:

$$C_l^{n+1} = C_l^n[i], \quad C_{l+1}^{n+1} = C_l^n[\omega] = 2C_l^n$$

and for l odd:

$$C_l^{n+1} = C_l^n[\omega] = 2C_l^n, \quad C_{l+1}^{n+1} = C_l^n[i].$$

For $n = 4m-1$ the formulas for l even and odd are reversed. For instance, $C^4 = 2C^3$, the straight sum of two quaternion fields or the algebra of dual quaternions; it is the system of elliptic biquaternions. C_1^4 is that of hyperbolic biquaternions. These results are related to those of D. Wajnsztejn [Ann. Soc. Polon. Math. 16, 65–83, 162–175 (1938); Studia Math. 9, 109–120 (1940); these Rev. 3, 103].

In the remaining part of the paper it is shown how to apply these results to establish a homoform correspondence of the groups O_l^n and S_l^n, and certain examples are given. It is concluded that all groups S_l^n are isomorphic to a subgroup of algebras of matrices with real complex, dual or dual complex elements. *D. J. Struik.*

Rund, Hanno. The theory of subspaces of a Finsler space. II. Math. Z. 57, 193–210 (1953).

This paper is a continuation of a previous one [Math. Z. 56, 363–375 (1952); these Rev. 14, 499] and gives further results on the theory of subspaces of a Finsler space. First, the author introduces the notion of relative curvature of

two curves C and C^* tangent at a point 0. It is defined by the consideration of two points $P \epsilon C$, $H \epsilon C^*$ at the same distance from 0 and by the study of the infinitesimal displacement HP. When C^* is a geodesic, the relative curvature is called the curvature of C. The rest of the paper is concerned with the study of hypersurfaces. The author takes the unit normal vector at a point P of a hypersurface F_{n-1}, displaces it parallelly to a neighboring point Q, and compares it with the unit normal vector at Q. By studying the difference of these two vectors, he is led to the notion of normal curvature of F_{n-1} at P for a direction x' and to a generalization of Meusnier's theorem. In terms of the normal curvature the second fundamental form is introduced. However, this second fundamental form has the "disadvantage" of being dependent on the direction. In order to retain the usual features of the theory of hypersurfaces of classical differential geometry, the author defines a secondary normal curvature and from that a secondary second fundamental form, which is a function of point only. With the use of the secondary second fundamental form, notions such as principal directions, Dupin indicatrix, conjugate directions, asymptotic directions, mean curvature, etc., can be defined. Various properties of these concepts are deduced. Finally, for later applications, the covariant derivative of the unit normal vector is computed in terms of the second fundamental form. *S. Chern.*

***Denis-Papin, Maurice, et Kaufmann, A. Cours de calcul tensoriel appliqué. (Géométrie différentielle absolue.)** Editions Albin Michel, Paris, 1953. 388 pp. 3440 francs.

Like two earlier works by the same authors [Cours de calcul opérationnel, Albin Michel, Paris, 1950; Cours de calcul matriciel appliqué, Albin Michel, Paris, 1951; these Rev. **11**, 593; **14**, 235], this book is intended for students of engineering. There are prefaces by F. Esclangon and G. Kron, the former containing a lively protest against conservatism in the mathematical education of engineers. The book is divided into two parts, nearly equal in length, the first dealing with the theory of tensors and the second with applications. In Part I the chapter headings are: I. Generalities before the theoretical study; II. Affine vector space; III. Euclidean vector space; IV. Tensor analysis in euclidean vector space; V. Riemannian spaces and geometry; VI. Use of matrices; and in Part II: VII. General dynamics; VIII. Dynamics of continuous media; IX. Tensor analysis of networks; X. Tensor theory of special relativity; XI. The electromagnetic field. There are two short appendices on positive definite quadratic forms and elementary notions on the theory of groups, a brief historical note, and a bibliography (11 pp.) of books and papers more or less related to the subject matter. There is a table of contents but no index. The book lays down the foundations of tensorial methods, and so should be useful for students of engineering and physics who wish to use such methods wherever applicable. In the applications, Chapter VIII contains a rather full account of finite displacements in elasticity. Chapter IX is in a sense the focus of the book; 69 pp. are devoted to an exposition of Kron's methods for electrical networks and machinery. The general style of writing is clear and lively, with details worked out explicitly and illustrative examples; exercises are attached to the chapters of Part I.
J. L. Synge (Dublin).

NUMERICAL AND GRAPHICAL METHODS

4 ***Thompson, Alexander John. Logarithmetica Brittannica, being a standard table of logarithms to twenty decimal places. Part II. Numbers 20,000 to 30,000 together with General Introduction.** Tracts for Computers No. XXII. Cambridge, at the University Press, 1952. 105 pp. (unpaged) +xcviii pp. +7 pp. +iii pp. 45 shillings; $8.50.

In 1624 Henry Briggs (1561–1631) published his great work Arithmetica logarithmica [Iones, London, 1624], containing $\log N$, $N=[1(1)20000, 90000(1)101000; 14D]$ and the square roots of integers $[1(1)200; 11D]$, with first differences in each case. To celebrate the tercentenary of this publication Karl Pearson arranged with his former student, A. J. Thompson, to prepare copy for the publication of Logarithmetica Brittannica to contain $\log N$, $N=[10000(1)100000; 20D]$. The first part, for $N=90000$–100000, was published in 1924, and eight parts had appeared by 1937. Now, fifteen years later, we have the ninth and final part with all necessary introductory material and title pages for binding the work into two volumes. The main table alone requires 900 pages. In this part II are 108 admirable preliminary pages, and in earlier parts was considerable matter of interest, especially in facsimiles of letters and of the will of Briggs, of title pages of books to which Briggs contributed material of importance, and the title page and pages of the excessively rare 1617 tract of Briggs: Logarithmorum Chilias Prima, the earliest publication of logarithms to the base 10. It exhibits the logarithms to 14D of the first thousand integers.

Thompson's manuscript for the main table carried $\log N$ to about 24D, the final digit being liable to an error of three or four digits. In the general introduction many pages are devoted to the method of construction of the table, to the integrating and differencing machine which Thompson built for his computations, to a description of how he himself set up all the type figures for publication of the table, and to a discussion (with examples) of interpolation. For each volume there is a factorizing Table F: $\log(1+N/10^7)$, $\log(1+N/10^{10})$, $\log(1+N/10^{13})$, $N=[0(1)1000; 21D]$. Table G has antilogarithms of logarithms 00000 00(1)0000 450; 21D. There is a page of "Short tables and constants" and a table of $\log N$, $N=[1(1)1000; 21D]$.

Six pages of errors in Briggs' Arithmetica logarithmica, 1624, are given and also an English translation of the earliest biography of Briggs in Thomas Smith's Vitae quorundam eruditissumorum et illustrium Virorum, 1707.

Here is a tremendous individual achievement, carried through with meticulous attention to every detail, in a manner most satisfying for any modern scholar or computer.
R. C. Archibald (Providence, R. I.).

***Yano, Tsuneta. Kokumin sûhyô. [People's tables.]** Kokuseisha, Tokyo, 1952. iv+146 pp. (1 plate). Yen 230.

These tables are intended for pencil and paper computation and include a multiplication table up to 100×100, small tables of reciprocals, powers, and roots, a factor table

from 1000 to 10000 and various small logarithm tables of which the most unusual is $\log N$ for $N=1(1)999$ to 15 decimals. *D. H. Lehmer* (Los Angeles, Calif.).

Mower, Lyman. Tables for second Born approximation scattering from various potential fields. Physical Rev. (2) 89, 947–949 (1953).

Variational methods for treating scattering problems in nuclear physics have been developed by Hulthén, Schwinger, and others. Using essentially Schwinger's approach, the author has calculated the scattering amplitude for electron scattering from various potentials, namely an exponential, a Yukawa, and a special combined potential. The quantities involved are tabulated as functions of energy and scattering angle, and the results are compared with those obtained from Born's first approximation. *P. O. Löwdin.*

Ishiguro, Eiichi, Hijikata, Katsunori, Arai, Tadashi, and Mizushima, Masataka. Tables useful for the calculation of the molecular integrals. I. Nat. Sci. Rep. Ochanomizu Univ. 1, 22–28 (1951).

Ishiguro, Eiichi, Arai, Tadashi, and Mizushima, Masataka. Tables useful for the calculation of the molecular integrals. II. Nat. Sci. Rep. Ochanomizu Univ. 2, 34–42 (1951).

Ishiguro, Eiichi, Arai, Tadashi, and Mizushima, Masataka. Tables useful for the calculation of the molecular integrals. III. Nat. Sci. Rep. Ochanomizu Univ. 3, 53–62 (1952).

These papers contain certain numerical tables useful in the study of molecular structures.

Part I contains: Table I, values to about 34 significant figures of e^x and e^{-x} for $x=.01(.01).1(.1)1(1)10$; Table II, values of

$$Z(m,n,j,k,p)$$
$$=(4\pi^2)^{-1}\iiiint\!\!\iint e^{-\alpha(\lambda_1+\lambda_2)}\lambda_1{}^m\lambda_2{}^n\mu_1{}^j\mu_2{}^k\rho^p d\lambda_1 d\lambda_2 \times d\mu_1 d\mu_2 d\varphi_1 d\varphi_2$$

for $\alpha=1.5$, $p=0$, $0\leqq m, n, j, k\leqq 8$.

Part II contains: Table III, values of $Z(m,n,j,k,p)$ for $\alpha=1.5$, $p=-1$, and similar values of m, n, j, k as in Table II; Table IV, values of

$$C_\tau{}^\nu(k)=\frac{1}{2}\int_{-1}^{1}P_\tau{}^\nu(\mu)(1-\mu^2)^{\frac{1}{2}\nu}\mu^k d\mu$$

for $\nu=0$, $0\leqq k$, $\tau\leqq 9$; Table V, values of $C_\tau{}^\nu(k)$ for $\nu=1$, $k=0(1)8$, $\tau=1(1)7$; Table VI, values of

$$\bar{C}_\tau{}^\nu(k)=C_\tau{}^\nu(k)-C_\tau{}^\nu(k+2)$$

for $\nu=0$, $k=0(1)6$, $\tau=0(1)5$; and Table VII, values of $\bar{C}_\tau{}^\nu(k)$ for $\nu=1$, $k=0(1)6$, $\tau=1(1)5$.

Part III contains: Table VIII, values of $C_\tau{}^2(k)$ for $k=0(1)7$, $\tau=2(1)7$; Table IX, values of $\bar{C}_\tau{}^2(k)$ for $k=0(1)6$, $\tau=2(1)7$; Table X, values of $C_\tau{}^\nu(j,k)=C_\tau{}^\nu(j)C_\tau{}^\nu(k)$ for $\nu=0$, $\tau=0(1)9$, and assorted j, k; Table XI, values for $C_\tau{}^1(j,k)$ similar to those in the preceding table; Table XII, similar values of $C_\tau{}^2(j,k)$; Table XIII, values of $\bar{C}_\tau{}^\nu(j,k)=\bar{C}_\tau{}^\nu(j)\bar{C}_\tau{}^\nu(k)$ for $\nu=0$; Table XIV, values of $\bar{C}_\tau{}^1(j,k)$; Table XV, values of $\bar{C}_\tau{}^2(j,k)$; Appendix I, exact expressions in terms of k and τ for $C_\tau{}^\nu(k)$, $\nu=0, 1, 2$; Appendix II, similar expressions for $\bar{C}_\tau{}^\nu(k)$; and corrections to Tables V, VI, VII. *A. Erdélyi* (Pasadena, Calif.).

***Bezicovici, I. S. Calcule approximative.** [**Approximate calculations.**] Editura Tehnică, Bucarest, 1952. 440+ii pp. Lei 21.46.

Romanian translation of Ya. S. Bezikovič's Približennye vyčisleniya, 6th ed. [Gostehizdat, Moscow, 1949].

Sobociński, Bolesław. On a universal decision element. J. Computing Systems 1, 71–80 (1953).

The author constructs a Boolean function of four variables $Q(p,q,r,s)$ from which may be obtained all sixteen Boolean functions of two variables by substitution of 0 and 1 for some of the variables, and by identification of variables. This function can be used as a model for the construction of a "decision element", that is, a unit of a digital computing machine, which is universal in the sense that all decision elements can be constructed from copies of it by making simple connections.

It is shown that no Boolean function of three variables has the same property. However, some Boolean functions of three variables called "quasi-universal" are exhibited, from which five of the ten non-trivial Boolean functions of two variables are obtainable by substitution of 0 and 1 and identification; the other five non-trivial functions are the negatives of the first five. *O. Frink.*

Sheldon, John, and Thomas, L. H. The use of large scale computing in physics. J. Appl. Phys. 24, 235–242 (1953).

Collatz, L. Aufgaben monotoner Art. Arch. Math. 3, 366–376 (1952).

In the search for simple and at the same time rigorous estimates of error, the author invites attention to a class of problems which he calls "problems of monotone type". In a space R of elements $u, v, \cdots, f, g, \cdots$ (real numbers, real vectors, real functions of real variables, etc.) consider an operator T which associates in single-valued manner the elements of a subspace B to the elements of a subspace D. The operator T need not be linear. If f is a given element of B, we seek in D a solution u of the equation $Tu=f$. This problem is of monotonic type if from the inequality $Tv\leqq Tw$ for arbitrary v, w in D it follows that $v\leqq w$. Clearly for such problems a rigorous error estimate is possible.

For examples to illustrate the general idea the author presents certain non-linear boundary-value problems in n real variables, boundary-value problems for ordinary linear differential equations of the second order, boundary-value problems for partial differential equations of second and fourth order, initial value problems for ordinary differential equations, and finally systems of linear algebraic equations. *W. E. Milne* (Corvallis, Ore.).

Fortet, R. Les méthodes de Monte-Carlo en physique nucléaire. Trabajos Estadistica 3, 341–371 (1952).

Probably the best and most complete single exposition of Monte Carlo "methods" that has yet appeared. The author develops the subject by describing the application to specific problems, with comments on advantages and shortcomings. These problems are: multi-dimensional integration; the diffusion of neutrons; the solution of the Fredholm integral equation; proper values of the Schroedinger equation. Various devices for reducing the variance of the estimate are discussed. While a more extensive bibliography might be helpful to the novice, he could hardly do better than to consult this paper as a start. *A. S. Householder.*

Vernotte, Pierre. Le rôle de la régularité dans le calcul numérique. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 45, xii+33 pp. (1952). 350 francs.

Arin' [Āriņš], E. G. On an application of graphical methods to nonlinear interpolation. Latvijas PSR Zinātņu Akad. Vēstis 1951, no. 1 (42), 154–162 (1951). (Russian. Latvian summary)

The author devises a graphical method for solving an elementary problem. *W. E. Milne* (Corvallis, Ore.).

Bruins, E. M. Square roots and cube roots. Nieuw Tijdschr. Wiskunde 40, 145–152 (1952). (Dutch)

If the number of digits of the divisor exceeds the capacity of the calculating machine, a method is applied which resembles the Babylonian method of division. The Babylonian method of calculating the square root can be appropriately modified in the case of "too long" divisors. The Heron method of cube roots is treated. Numerical examples illustrate the exposition. *H. Freudenthal* (Utrecht).

Myrhøj, A. M. S. On root extraction. Mat. Tidsskr. A. 1952, 22–40 (1952). (Danish)

Stone, W. M. A form of Newton's method with cubic convergence. Quart. Appl. Math. 11, 118–119 (1953).

Il existe des formules du type de Newton avec convergence cubique [Stewart, Amer. Math. Monthly 58, 331–334 (1951); Hamilton, ibid. 57, 517–522 (1950); ces Rev. 12, 537; Bodewig, Quart. Appl. Math. 7, 325–333 (1949); ces Rev. 11, 136]. Ces diverses formules contiennent des dérivées secondes de la fonction. L'auteur en propose une qui consiste à remplacer la dérivée seconde par la différence de deux valeurs de la dérivée première. *J. Kuntzmann.*

Blaquière, A. Adaptation générale de la méthode du diagramme de Nyquist dans le domaine non linéaire. J. Phys. Radium (8) 13, 636–644 (1952).

A one-parameter family of linear approximations to the given non-linear equation is obtained. And a study of the varying diagrams for these leads to the desired results for the given non-linear system. *P. Franklin.*

Štykan, A. B. Graphical computation of Stieltjes integrals. Doklady Akad. Nauk SSSR (N.S.) 87, 893–895 (1952). (Russian)

Having in a previous paper [Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 184–191 (1952); these Rev. 13, 992] given the graphical solution of differential equations with a deviated argument, the author turns now to a Stieltjes integral with deviated argument of the type

$$y(x)=\int_{x_0}^{x} F[x, y(x), y(x\pm\alpha(x)), y(x\pm\beta(y))]d[\zeta(\tau(x))].$$

This equation is solved graphically in much the manner as in the case of the differential equations of the earlier article. *W. E. Milne* (Corvallis, Ore.).

Štykan, A. B. Graphical methods of solution of some problems of mathematical analysis. Akad. Nauk SSSR. Inženernyĭ Sbornik 13, 177–186 (1952). (Russian)

By means of examples the author shows how to solve graphically such problems as the integration of functions, solution of ordinary differential equations of first order, systems of such equations, finding an approximate mean value for a derivative, etc. He uses a method called "superposition of coordinates", and another called the "method of straight lines". *W. E. Milne* (Corvallis, Ore.).

Matthieu, P. Über die Fehlerabschätzung beim Extrapolationsverfahren von Adams. II. Gleichungen zweiter und höherer Ordnung. Z. Angew. Math. Mech. 33, 26–41 (1953). (English, French and Russian summaries)

A continuation and extension to equations of higher order of the author's earlier paper [same Z. 31, 356–370 (1951); these Rev. 13, 691]. The methods are similar to those of the first paper, and again emphasis is placed on putting results in a form for practical computation. *W. E. Milne.*

Weissinger, J. Numerische Integration impliziter Differentialgleichungen. Z. Angew. Math. Mech. 33, 63–65 (1953).

Methods for the numerical solution of differential equations have mainly been devised for explicit equations such as $y'=f(x, y)$. The author undertakes the numerical solution of the implicit equation $F(x, y, y')=0$. Naturally this involves not only step-by-step integration but also the solution of the implicit equation at each step. The first part of the paper is devoted to presenting practical ways for carrying out the numerical work of solving and integrating with the aid of difference tables. Formulas are given for an estimate of the error. *W. E. Milne* (Corvallis, Ore.).

Pailloux, Henri. Une méthode d'approximation. C. R. Acad. Sci. Paris 236, 1133–1134 (1953).

The author describes a modification of the standard Rayleigh-Ritz procedure which may be used to determine approximately the solution of a variational problem. His method is intended to remove the difficulty which arises from trying to satisfy the boundary conditions. The boundary values of the original problem are incorporated in some appropriate functional whose square integral over the boundary should be required to be a minimum. It is proposed that of the n Rayleigh-Ritz coefficients some should be determined by the requirement that the new boundary condition be satisfied, while the remainder should be determined by the condition that the original integral have an extreme value. The note is brief and does not contain any illustrative examples. Convergence is not claimed for the method in general, but examination of successive approximations is suggested as a means for getting a practical idea of the validity of the results for any particular problem. *E. Isaacson* (New York, N. Y.).

★v. Mises, R. On network methods in conformal mapping and in related problems. Construction and applications of conformal maps. Proceedings of a symposium, pp. 1–5. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

An exposition, with carefully selected examples, of some of the main difficulties involved in solving boundary value problems by network methods. A large number of unsolved problems in this area are specified, concerning especially: the accuracy of solutions, the choice of the network, and getting higher order approximations. *G. Birkhoff.*

★Rosenbloom, P. C. The difference equation method for solving the Dirichlet problem. Construction and applications of conformal maps. Proceedings of a symposium, pp. 231–237. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The author gives an explicit solution of the finite difference equations which approximate Poisson's equation in a

rectangular domain. He also obtains an estimate for the difference between the true solution and the solution of the finite difference equations. *B. Friedman.*

Synge, J. L. Triangulation in the hypercircle method for plane problems. Proc. Roy. Irish Acad. Sect. A. **54**, 341–367 (1952).

In the method of the hypercircle as introduced by Prager and Synge [Quart. Appl. Math. **5**, 241–269 (1947); these Rev. **10**, 81] one approaches the solution of a boundary-value problem derivable from a variational principle by splitting all conditions of the problem into two parts, A and B. The two "relaxed" problems, A and B, each possess an infinity of solutions and the solution of the original problem is the intersection of the two infinite sets. One of the main problems in this method is the selection of suitable approximation sequences for the solution of both problems A and B such that convergence toward the solution of the original problem can be guaranteed. In the present paper the author introduces a system of functions suitable for this purpose in the case of two independent variables. These functions are piecewise linear and are defined with reference to a triangular net in the plane of the independent variables. Continued refinement of the net with resulting increase of the number of the "pyramid functions" is said to lead to convergence. As an application a problem for a membrane with hexagonal boundary is considered. *E. Reissner.*

***Austin, W. J., and Newmark, N. M. A numerical method for the solution of plate buckling problems.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 363–371. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Jaeger, J. C. Conduction of heat in a solid with periodic boundary conditions, with an application to the surface temperature of the moon. Proc. Cambridge Philos. Soc. **49**, 355–359 (1953).

The author develops a technique for calculating heat conduction in a solid with temperature distribution at the boundary which is a periodic function of time. The problem is treated by dividing the period T into N equal intervals and using a temperature distribution which is a periodic step function. The author points out that other approximations, such as a linear function in each interval, may also be used. Application is made to the variation of the surface temperature of the moon at the equator during a lunation. *H. Polachek* (Cardarock, Md.).

Schneider, P. J. Numerical method for porous heat sources. J. Appl. Phys. **24**, 271–274 (1953).

McFadden, J. A. Summation of Fourier series. J. Appl. Phys. **24**, 364 (1953).

Milatz, J. M. W., Wapstra, A. H., and Van Wieringen, J. S. A simple method for a Fourier analysis. Physica **19**, 175–180 (1953).

Young, F. H. The NOTS REAC. Amer. Math. Monthly **60**, 237–243 (1953).

Ludeke, Carl A., and Morrison, Cohn L. Analog computer elements for solving nonlinear differential equations. J. Appl. Phys. **24**, 243–248 (1953).

McDonal, F. J. A Fourier analyzer. Rev. Sci. Instruments **24**, 272–276 (1953).

Malavard, Lucien, et Boscher, Jean. Modèle analogique électrique pour l'étude de la flexion des poutres. C. R. Acad. Sci. Paris **236**, 1130–1133 (1953).

Borden A., Shelton, G. L., Jr., and Ball, W. E., Jr. An electrolytic tank developed for obtaining velocity and pressure distributions about hydrodynamic forms. The David W. Taylor Model Basin, Washington, D. C. Rep. **824**. iv+29 pp. (1953).

Pajares, Emilio. On a problem in the determination of orbits of double stars. Revista Acad. Ci. Madrid **46**, 307–313 (1 plate) (1952). (Spanish)

Because of errors of observation, the measured time-distance relation $\rho(t)$ and the time-angle relation $\theta(t)$ of the orbits of double stars do not satisfy the Kepler condition $\rho^2\theta' = C$ that the areal velocity of the radius vector be constant. The measured values are smoothed and adjusted by means of a proposed mechanism. A value of C is selected. A pantagraph mounted on a carriage is adjusted. A bar, one end of which slides in a slot and traces the time-distance curve, is kept tangent to a hyperbolic cam on the carriage. Then, a scriber on the carriage draws the time-angle curve. *M. Goldberg* (Washington, D. C.).

ASTRONOMY

Nadile, Antonio. Configurazioni ellissoidali di equilibrio di una massa liquida omogenea attratta da un anello circolare concentrico. Atti Sem. Mat. Fis. Univ. Modena **5**, 178–189 (1951).

The existence of a system composed of a central body in the form of an ellipsoid of Jacobi or Maclaurin and of a concentric ring is derived under the condition that the radius (u) of the ring is very large and its cross-section very small. Putting δ for the maximum dimension in the central body, the terms of the order δ^2/u^3 are neglected in the potential of the circle representing the ring. The necessary and sufficient condition for the existence of such a configuration is given in the form similar to that for ellipsoids of Jacobi. *W. S. Jardetzky* (New York, N. Y.).

Graffi, D. Sul problema dei due corpi di massa variabile. Ann. Univ. Ferrara. Sez. VII (N.S.) **1**, 23–33 (1952).

This paper is a continuation of some previous papers by the same author [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **68**, 262–272, 459–482 (1933); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **19**, 144–151, 228–233 (1934); Ann. Mat. Pura Appl. (4) **15**, 87–128 (1936)]. A new upper bound is determined for the variations of the eccentricity of the osculating orbit in the problem of two bodies with variable masses. This new upper bound is found to be particularly suitable in the case of small eccentricities. In addition, a confirmation of a previous theorem to the effect that small variations of the eccentricity are not incompatible with large variations of the masses is obtained. *E. Leimanis* (Vancouver, B. C.).

Volpato, Mario. Un'osservazione sulle approssimazioni della soluzione del problema dei due corpi di massa variabile. Ann. Univ. Ferrara. Sez. VII (N.S.) 1, 35–46 (1952).

An approximate solution is given for the so-called reduced problem of two bodies with variable mass [cf. Armellini, Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 25, 297–301 (1937)]. This is the case where it is assumed that the eccentricity of the osculating orbit is infinitely small. Making use of a recent result obtained by Graffi [see the preceding review], it is shown that in the astronomical case (when the total mass varies slowly with the time) the approximate solution obtained differs little from the exact solution, and holds under less restrictive conditions than the approximate solution obtained by Michelacci [Pont. Acad. Sci. Comment. 8, 1–12 (1944); these Rev. 10, 332], using a different method. This result also confirms the legitimacy of replacing the equation, governing the motion of two bodies with variable mass, by the equation of Armellini in the case of the reduced problem.

E. Leimanis (Vancouver, B. C.).

Bélorizky, D. Sur la régularisation du problème des trois corps par la variable de Sundman. Bull. Astr. 16, 327–331 (1952).

In order to regularize the singularity of the differential equations of motion at a double collision of the bodies, K. F. Sundman [Acta Math. 36, 105–179 (1912)] introduced the new independent variable ω defined by

$$dt=[1-\exp(-r_1/p)][1-\exp(-r_2/p)]\times[1-\exp(-r_3/p)]d\omega,$$

where r_1, r_2, r_3 are the three mutual distances of the bodies and p is the lower bound of the two greater distances. The variable ω regularizes the motion within a band of finite breadth in the ω-plane, bounded by two lines parallel to the real axis and at the same distance on either side of it. The author examines the question as to whether the variable ω is capable of regularizing the motion in the whole ω-plane. The consideration of the Lagrangean equilateral solutions shows that neither ω nor the variable φ defined by $dt=r_1r_2r_3d\varphi$ is capable of doing so. Consequently neither of these variables can completely regularize the problem of three bodies [cf. Bélorizky, C. R. Acad. Sci. Paris 231, 1428–1429 (1950); these Rev. 12, 550]. *E. Leimanis.*

Grémillard, Jean. Sur certaines solutions périodiques du problème des trois corps. C. R. Acad. Sci. Paris 236, 49–51 (1953).

This paper is a continuation of an earlier one [same C. R. 234, 2339–2341 (1952); these Rev. 13, 996] in which the author showed the existence of periodic solutions of the third kind, the initial eccentricities being zero and the initial inclinations being entire series in terms of an auxiliary variable ρ. The condition for their existence is that the mean motions n and n' of the two osculating trajectories satisfy the relation $n/n'=p/q$, where p and q are relatively prime positive integers and $p-q$ is an even number at least equal to 4. Assuming that $p-q$ is an odd integer greater than one, the author shows the existence of periodic solutions of the third kind corresponding to initial eccentricities and initial inclinations of the orders $p-q-1$ and one with respect to ρ, respectively. For $\rho=0$ these expansions lead to periodic solutions of the first kind whether $p-q$ is an even or odd number. *E. Leimanis* (Vancouver, B. C.).

Meffroy, Jean. Sur les termes séculaires du développement des grands axes par rapport aux masses. C. R. Acad. Sci. Paris 236, 778–780 (1953).

The celebrated theorem of Poisson on the perturbations of the semi-major axes of planetary orbits states that, if these perturbations are developed in powers of the masses, the terms of the second power in the masses are periodic or mixed secular; no purely secular terms are present. The result applies only if no commensurabilities arise.

This paper treats the perturbations of the third power in the masses. It is shown by explicit developments that terms arise that are purely secular. The result agrees with that arrived at by S. C. Haretu [Ann. Observ. Paris. 18, I, 1–39 (1885)] and by D. Eginitis [ibid. 19, H, 1–16 (1889)], but it is shown that in these earlier derivations incomplete expressions had been used. *D. Brouwer.*

Davis, Morris S. A study of the method of rectangular coordinates in general planetary theory. Astr. J. 56, 188–199 (1952).

Detailed developments are given of functions left in symbolic form in Brouwer's theory of general planetary perturbations in rectangular coordinates [Astr. J. 51, 37–43 (1944); these Rev. 6, 189]. The properties of these functions are compared with similar functions arising in Hansen's form of planetary theory. First-order perturbations of the minor planet (185) Eunike derived by the two methods are presented and compared. With a few exceptions, the agreement is within a few units in the sixth decimal place. The evaluation of the constants of integration is then treated, and the paper concludes with a comparison of the efficiency of the two methods as adapted to modern computing machine techniques. *D. Brouwer* (New Haven, Conn.).

Eichhorn, Heinrich. Die Genauigkeit einer Kreisbahnbestimmung. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 160, 251–266 (1951).

If the spherical coordinates λ_i, β_i, observed at times t_i $(i=1, 2)$, are used for the determination of a circular orbit, the deduced elements will be affected by the observational errors present in the coordinates. Let E designate any one of the four elements of the orbit. The evaluation of the dependence of an element on the observed coordinates may be obtained with the aid of $\partial E/\partial\lambda_i$, $\partial E/\partial\beta_i$. The derivation of expressions for these partial derivatives is the principal object of this paper. The case of the circular orbit is chosen on account of its simplicity compared with the general case of the elliptic orbit. *D. Brouwer* (New Haven, Conn.).

Dilgan, Hâmit. Sur la vitesse moyenne des planètes. Bull. Tech. Univ. Istanbul 4 (1951), no. 1, 21–24 (1952). (Turkish summary)

The energy integral of the two-body problem gives for the linear velocity in the orbit

$$v=\mu^{\frac{1}{2}}\left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{1}{2}}=\phi(r).$$

To find the mean value of v, the author evaluates the integral

$$\frac{1}{r_{\max}-r_{\min}}\int_{r_{\min}}^{r_{\max}}\phi(r)dr.$$

[The result has no obvious physical meaning as this is not the time average of the orbital velocity.] *D. Brouwer.*

★**Ambarcumyan, V. A., Mustel', È. R., Severnyi, A. B., i Sobolev, V. V. Teoretičeskaya astrofizika.** [**Theoretical astrophysics.**] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 635 pp. (1 plate). 15 rubles.

Contents: I. Theory of radiative equilibrium of stellar photospheres and continuous spectrum of stars: Radiative transfer theory; Coefficient of continuous absorption; Structure of stellar photospheres; Thermodynamic equilibrium. II. Formation of absorption lines in stellar spectra: Mechanism of absorption line formation; Solution of equation of transfer for absorption line frequencies; Coefficients of selective absorption; Absorption line contours; Curve of growth; Chemical composition; Spectral sequences; Scale of effective temperatures. III. Physics of the solar envelope: Structure of the photosphere; Granules; Electrodynamics of the solar atmosphere; Sunspots; Faculae; Prominences; Flocculi; Flares; The corona; Solar radio noise. IV. Planetary nebulae: Temperature of nucleus; Physical state; Radiative equilibrium. V. Novae. VI. Stars with bright lines: Formation of emission lines. VII. Interior structure of stars: Physical state. VIII. Dispersion of light in the planets' atmospheres: Radiative transfer; Albedo. IX. Interstellar matter: Diffuse nebulae; Interstellar gas.

This book, designed for a graduate course, is (after the introductory pages on radiation theory) in the main a compilation of recent results of Soviet astrophysicists with special attention to investigations made by the authors themselves. Thus there is no or next to no material on variable stars, binaries, comets, stellar evolution, solar system origin, and interstellar radio noise—topics not studied intensively by the Russians. The authors also express disfavor for stellar models, with the result that the space devoted to stellar interiors is held to a minimum.

In none of the fields covered is the theory much developed. Rather, despite the word "theoretical" in the title, all effort is expended toward obtaining practical results as quickly as possible. *R. G. Langebartel* (Urbana, Ill.).

Neyman, J., and Scott, E. L. A theory of the spatial distribution of galaxies. Astrophys. J. 116, 144–163 (1952).

In this paper a theory of the spatial distribution of galaxies is developed on the basis of the following four assumptions: (1) galaxies occur only in clusters; (2) the number of galaxies varies from cluster to cluster, subject to a probabilistic law; (3) the distribution of galaxies within a cluster is also subject to a probabilistic law; and (4) the distribution of cluster centers in space is subject to a probabilistic law described as quasi-uniform. The main result obtained is the joint probability generating function $G_{N_1,N_2}(t_1,t_2)$ of numbers N_1 and N_2 of galaxies visible on photographs from two arbitrarily placed regions ω_1 and ω_2, taken with fixed limiting magnitudes m_1 and m_2, respectively.

The paper contains a number of new results in mathematical statistics. An example is the following. From the postulates (a) to every region R with volume $\mathfrak{B}$ there corresponds a random variable $\gamma(R)$ representing the number of cluster centers falling within R [the distribution of $\gamma(R)$ depends on the volume $\mathfrak{B}$ only] and (b) whatever be the non-overlapping regions $R_1, R_2, R_3, \cdots$, the corresponding random variables $\gamma(R_j)$, $j=1, 2, \cdots$, are completely independent, it follows that the probability generating function $G_\gamma(t|\mathfrak{B})$ of $\gamma(R)$ has the form

$$G_\gamma(t|\mathfrak{B})=[G_\gamma(t|1)]^{\mathfrak{B}} \tag{1}$$

where

$$G_\gamma(t|1)=\exp\left\{-h_0+\sum_{k=1}^{\infty}h_kt^k\right\}. \tag{2}$$

In (2), $h_0, h_1, \cdots, h_k, \cdots$ are all nonnegative numbers subject to the restriction that the series $\sum_{k=1}^{\infty}h_k$ is convergent and that $\sum_{k=1}^{\infty}h_k=h_0$. *S. Chandrasekhar.*

Chandrasekhar, S. A statistical basis for the theory of stellar scintillation. Monthly Not. Roy. Astr. Soc. **112**, 475–483 (1952).

The point of departure of this investigation is the assumption that the cause of atmospheric scintillation is a turbulent layer in the upper troposphere, in which the refractive index is subject to irregular fluctuation. The author defines the auto-correlation of the instantaneous fluctuations in the refractive index at two different points, and on the assumption that the turbulence is homogeneous and isotropic shows that the auto-correlation is a definite function of the distance between the two points considered. He further shows how the statistical properties of the corrugated emergent wavefront, such as the angular dispersion in the wave normals, can be expressed in terms of this function. From the known facts concerning astronomical seeing the author concludes that all essential phenomena can be satisfactorily accounted for if we postulate the existence of a turbulent atmospheric layer, at a height of about 4 km, of a thickness of the order of a hundred metres, exhibiting turbulence of a micro-scale of the order of ten centimeters and a root-mean-square fluctuation in refractive index of the order of 4×10^{-8}. *Z. Kopal* (Manchester).

Limber, D. Nelson. The analysis of counts of the extragalactic nebulae in terms of a fluctuating density field. Astrophys. J. 117, 134–144 (1953).

In this paper a method is developed for analyzing the counts of the extragalactic nebulae in terms of a fluctuating density field for the nebulae in space. It is shown how such an analysis can lead to a determination of the parameters characterizing the distribution of the nebulae; and that the effects of galactic absorption can be taken into account on the recent theory of Chandrasekhar and Münch [same J. **115**, 103–123 (1952); these Rev. **13**, 786] for the fluctuations in brightness of the Milky Way. The necessary expressions for analyzing the counts are developed for four separate cases so that the effects of a dispersion in absolute magnitudes of the nebulae and of fluctuations in galactic absorption can be either included or neglected as the situation may demand. *Z. Kopal* (Manchester).

Limber, D. Nelson. Fluctuations in brightness of the Milky Way. Astrophys. J. 117, 145–168 (1953).

The object of this paper is an extension of the Chandrasekhar-Münch theory of fluctuations in the brightness of the Milky Way in terms of a fluctuating field of absorbing matter [same J. **115**, 103–123 (1952); these Rev. **13**, 786] to systems of finite extent. The theoretical extensions necessary for an analysis of observational material are developed, and the analysis of the Pannekoek's isophotal charts of the southern Milky Way is carried out. The latitude dependence of the brightness correlations given by the theory is found to give satisfactory agreement with the observations for a microscale of the fluctuations of 0.01 in units of the mean optical thickness; a root-mean-square of the fluctuations in

density of $5\bar{\rho}$; and an "effective" half-thickness of the relevant galactic absorbing matter of 0.20 magn. *Z. Kopal.*

Horak, Henry G. The transfer of radiation by an emitting atmosphere. Astrophys. J. 116, 477–490 (1952).

The transfer of radiation by a plane-parallel stellar atmosphere containing a uniform distribution of emission sources is considered for two cases: (a) isotropic scattering with an albedo $\tilde{\omega}$; and (b) scattering according to the asymmetric phase function $\tilde{\omega}(1+x\cos\theta)$, where θ denotes the angle between incident and scattered beams and x is a constant. The corresponding equations of transfer are set up and solved by the method of discrete ordinates; it is pointed out that exact expressions for the emergent intensity can be obtained both for the finite and semi-infinite atmospheres. Numerical applications are carried out for the emergent radiation for a semi-infinite emitting atmosphere in the case (a) with $\tilde{\omega}=0.5$; and in the case (b) for $\tilde{\omega}=x=0.5$. Further calculations have been made for finite atmospheres at the optical depth $\tau=0.20$. These formulae are also applied to the case of the galactic light (due to the scattering of starlight by interstellar material). *Z. Kopal* (Manchester).

Plass, Gilbert N., and Fivel, Daniel I. Influence of Doppler effect and damping on line-absorption coefficient and atmospheric radiation transfer. Astrophys. J. 117, 225–233 (1953).

The author gives the form of both the general Taylor series for the line-absorption coefficient in powers of $a=(\alpha/\Delta\nu_D)(\Delta\nu_D$ being half the Doppler width at half-maximum, and α the half-Lorentz plus natural width at half maximum), as well as an asymptotic expansion in inverse powers of $\omega=(\nu-\nu_0)/\Delta\nu_D$ when both the Doppler effect and damping contribute to the line shape. The influence of the Doppler effect on the fractional absorption of a line as measured in the laboratory is calculated, and compared with expressions which assume a Lorentz line shape. The radiation transfer in a planetary atmosphere is calculated for the case in which both the Doppler effect and damping contribute to the line shape; and the change of half-width with height is taken into account. It is shown that, for the Earth's atmosphere, the radiative transfer calculated from the Lorentz shape alone is not changed appreciably by the Doppler effect for either weak or strong lines at heights up to at least 50 km, even if the Doppler width is somewhat greater than the Lorentz width.
Z. Kopal (Manchester).

Huang, Su-Shu. On the absorption lines formed in stellar atmospheres. Ann. Astrophysique 15, 169–183 (1952).

The introductory section of this paper contains an outline of the mathematical theory of the formation of absorption lines in stellar atmospheres. In the second section, the Schwarzschild integral equation of transfer is set up, and an elementary (iterative) method is given for proving the convergence of the Liouville-Neumann expansion of its solution. In physical terms, the iteration method used here is equivalent to taking account of the radiation of successive scattering; thus the first approximation gives a purely thermal radiation field (with scattering completely neglected); the second approximation includes (besides radiation coming directly from thermal emission) all radiation scattered once; etc. In the third section of this paper, the form of the line profiles is derived from the flux observations; while in section four an alternative derivation of line profiles from the intensity observations is outlined. The concluding section five contains a comparison of the author's results with the earlier Milne-Eddington model and some concluding remarks. *Z. Kopal* (Manchester).

Gjellestad, Guro. Magneto-hydrodynamic oscillations of a star. Ann. Astrophysique 15, 276–289 (1952).

Considering non-radial oscillations of an incompressible fluid sphere which deform the boundary in the manner $r(\mu)=R+\epsilon(t)P_l(\mu)$ where R denotes the unperturbed radius, μ the cosine of the polar angle, $P_l(\mu)$ the Legendre polynomial of order l and $\epsilon/R\ll 1$, Kelvin showed that the characteristic frequencies of the oscillations are $[8\pi G\rho l(l-1)/3(2l+1)]^{1/2}$. The corresponding velocity field during the oscillations is given by

$$(1)\quad v_r=\left(\frac{r}{R}\right)^{l-1}P_l(\mu)\frac{d\epsilon}{dt}\quad\text{and}\quad v_\theta=-\frac{1}{l}\left(\frac{r}{R}\right)^{l-1}P_l^1(\mu)\frac{d\epsilon}{dt},$$

where the subscripts r and θ refer to the radial and the θ-components of $\mathbf{v}$. If we now suppose that the sphere is of infinite electrical conductivity and is imbedded in a magnetic field of intensity $\mathbf{H}$ which is sufficiently small for its effects on the motion to be negligible, then we may determine the variation of the magnetic field during oscillation from

$$(2)\quad \frac{\partial}{\partial t}\delta\mathbf{H}=\operatorname{curl}(\mathbf{v}\times\mathbf{H})$$

which follows from Maxwell's equations. When the initial field is uniform and of intensity H in the z-direction, the foregoing relation gives

$$\delta H_r=H(l-1)\frac{r^{l-2}}{R^{l-2}}P_{l-1}(\mu)\epsilon$$

(3) and

$$\delta H_\theta=-H\frac{r^{l-2}}{R^{l-1}}P^1{}_{l-1}(\mu)\epsilon.$$

This is the essential substance of the author's paper, although the results are not derived in this manner.
S. Chandrasekhar (Williams Bay, Wis.).

Helfer, H. Lawrence. Magneto-hydrodynamic shock waves. Astrophys. J. 117, 177–199 (1953).

The author considers the propagation of plane waves, of finite amplitude, in an infinitely conducting medium in a magnetic field, on the basis of the de Hoffman-Teller shock-wave conditions (which represent a generalization of the Rankine-Hugoniot shock-wave conditions of ordinary hydrodynamics of non-conducting fluids). Two specific consequences of the de Hoffman-Teller theory are discussed numerically in some detail. Perhaps the most striking result of this investigation from the astrophysical point of view is that fact that, in the process of collision of interstellar clouds, weak magnetic fields which may be initially present are always amplified by the passage of shock fronts; and if suitable external mechanisms are available to reduce the compressional effects of shock waves, the magnetic field may grow steadily until it reaches the value of $(8\pi p)^{\frac{1}{2}}$ gauss, where p stands for the pressure. The second application concerns the ionized gas at the conditions approximating those in the inner solar corona; and it is shown that general magnetic fields of strengths of the order of 0.1 gauss lead to results which are not inconsistent with the observed internal motions of the solar prominences. *Z. Kopal.*

RELATIVITY

★**Einstein, Albert. The meaning of relativity.** 4th ed. Princeton University Press, Princeton, N. J., 1953. iv+168 pp. $3.50.

Since the main text and Appendix I remain unchanged in this edition, only Appendix II is discussed in this review. For reviews of the 2d and 3d editions see these Rev. 7, 87; 14, 97.

The present form of Einstein's generalization of gravitation theory differs from those proposed in earlier editions of this book in that the field equations are derived from a variational principle. The theory has to do with an affine space with a connection $\Gamma^i{}_{jk}$ which is not symmetric. In this space there is a non-symmetric tensor g_{ij}, the symmetric part of which is to represent gravitation and the anti-symmetric part is to be related to the electromagnetic field. The field equations are derived from the variational principle,

$$\delta\int L\,d\tau=0,$$

where the Lagrangean function L is a function of the g_{ij} and $\Gamma^i{}_{jk}$ and these quantities are to be varied independently. The function L is chosen so that one of the variational equations shall be

$$(\sqrt{-g}\,g_{\underset{+\,-}{ik}})_{;l}\equiv(\sqrt{-g}\,g_{ik})_{,l}-\sqrt{-g}\,g_{sk}\Gamma^s{}_{il}$$
$$-\sqrt{-g}\,g_{is}\Gamma^s{}_{lk}-\sqrt{-g}\,g_{ik}\Gamma^s{}_{\underline{ls}}=0$$

where $g=\det\|g_{ik}\|$, the comma denotes ordinary differentiation, and the line under a set of subscripts denotes the symmetric part of the tensor whose subscripts are underlined.

Another condition on the choice of the Lagrangean function is that the resulting field equations shall be (Hermitean) such that they are invariant when $g_{ik}\to\tilde{g}_{ik}=g_{ki}$ and $\Gamma^j{}_{ik}\to\tilde{\Gamma}^j{}_{ik}=\Gamma^j{}_{ki}$. Einstein finds such a Lagrangean:

$$\sqrt{-g}\,g^{ij}R^{\bullet\bullet}_{ij}=\sqrt{-g}\,g^{ij}\left(R_{ij}+\Gamma_{\underset{+}{i};k}-\left(\frac{1}{\sqrt{-g}}(\sqrt{-g})_{,\underset{+}{i}}\right)_{;k}\right)$$

where R_{ij} is the Ricci tensor formed from the $\Gamma^i{}_{jk}$ and

$$\Gamma_i=\Gamma^s_{\underset{\smile}{is}},$$

the symbol $\smile$ under a pair of indices denoting the anti-symmetric part of the tensor so marked.

In the final field equations the g_{ij} are not varied independently but are subject to the restriction that

$$(\sqrt{-g}\,g^{\underset{\smile}{is}})_{,s}=0.$$

A notion of strength of field equations is introduced, discussed, and used to justify restricting the variations of the g_{ij} in this way. The final field equations obtained are:

$$(\sqrt{-g}\,g_{\underset{+\,-}{ik}})_{;l}=0,\qquad \Gamma_i=0,$$
$$R_{ik}=0,\quad R_{\underset{\smile}{ik},l}+R_{\underset{\smile}{kl},i}+R_{\underset{\smile}{li},k}=0.$$

The author closes Appendix II with a set of "General remarks concerning the concepts and methods of theoretical physics". In this section he reviews the role of the field concept in physics and its connection with a continuum-space-time. He expresses his convictions concerning the necessity for pure field theory and the fact that this must arise from a generalization of the theory of gravitation for empty space. *A. H. Taub* (Urbana, Ill.).

★**Einstein, Albert. Sur le problème cosmologique. Théorie de la gravitation généralisée.** Traduit de l'anglais par Maurice Solovine. Gauthier-Villars, Paris, 1951. 54 pp. 500 francs.

The first part is translated from the appendix to the 2d edition of "The meaning of relativity" [Princeton Univ. Press, 1945; these Rev. 7, 87] (Appendix I of the 3d and 4th editions (1950, 1953). The second part is a translation of Appendix II of the 3d edition [Princeton Univ. Press, 1950; these Rev. 14, 97], but with considerable revision. It is distinct from the version of Appendix II in the 4th edition reviewed above.

Kurşunoğlu, Behram. Gravitation and electrodynamics. Physical Rev. (2) **88**, 1369–1379 (1952).

A new version of Einstein's unified field theory is given. The equations proposed involve a "fundamental constant" of the dimension of (length)$^{-1}$. The author obtains 18 equations for 16 field variables ($g_{\alpha\beta}\neq g_{\beta\alpha}$) plus two trivial identities. The conservation laws are discussed; the 4-current distribution is obtained, as well as the energy-momentum pseudo-tensor of the total field. In the linear approximation the electrostatic field and the static charge density is calculated. In this approximation the electromagnetic field can be split up into a short-range and a long-range part. The classical description of pair production and the action function of Maxwell-Lorentz theory are also obtained. Deflection of light in this theory is the same as that given by the general theory of relativity. *R. Teisseyre* (Warsaw).

★**Narlikar, V. V. From general relativity to a unified field theory.** Proceedings of the Fortieth Indian Science Congress, Lucknow, 1953, Part II, Section 1, pp. 1–20. Indian Science Congress Association, Calcutta, 1953.

A brief description is given of general relativity, particularly as regards the solution of Einstein's equations in free space and of the meaning of singularities in these solutions. The unified field theories of Bergmann, Schrödinger, and Einstein are then described and their shortcomings noted. None of these theories have solved as yet any physical problem and they run counter to the ideas of quantum mechanics. The paper contains many pertinent and important comments on relativity theories, which are too numerous to mention individually. *G. C. McVittie.*

Goldberg, Joshua N. Strong conservation laws and equations of motion in covariant field theories. Physical Rev. (2) **89**, 263–272 (1953).

Covariant field theories possess Bianchi identities and hence strong conservation laws, i.e., certain ordinary divergences vanish whether or not the field equations are satisfied. Vanishing of divergences enables one to introduce "superpotentials" by means of which two-dimensional surface integrals can be written which yield the equations of motion for the singularities enclosed [Einstein and Infeld, Canadian J. Math. 1, 209–241 (1949); these Rev. 11, 59]. In the Hamiltonian formalism the second time derivative of the primary constraints [J. Anderson and P. G. Bergmann, Physical Rev. (2) **83**, 1018–1025 (1951); these Rev. 13, 411] leads to the Bianchi identities in terms of canonical variables. Hence follows the existence of strong conservation laws and of superpotentials in the Hamiltonian formalism. *M. Suffczynski* (Warsaw).

Bergmann, Otto. Zur Optik in der verallgemeinerten Feldtheorie. Acta Physica Austriaca 6, 306–318 (1953).

The author discusses the solutions of the Schrödinger generalized field theory under the assumption that the gravitational potentials are those corresponding to the metric of Minkowski space and the electromagnetic field quantities correspond to a constant and uniform field plus small time- and space-dependent terms which represent the propagation of light. The constant field produces anisotropy of space-time and the effect of this anisotropy on the propagation of light is discussed. *A. H. Taub* (Urbana, Ill.).

Takasu, Tsurusaburo. Sphere-geometrical unitary field theories. Compositio Math. 10, 95–116 (1952).

The main scope of this paper is the four-dimensional interpretation of different n-dimensional unified theories of relativity ($n=5, 6$) by means of Laguerre's spherical geometry. The main result may be formulated as follows: The n-dimensional world-points are realized within the Einstein space V_4 by means of oriented geodesic spheres, oriented linear hypercomplexes, geodesic hyperspheres, and so on. The method of proofs is based on the interpretation of the corresponding metric form by means of the spherical geometry as developed by the author in his books, Differentialgeometrien in der Kugelräumen [Bd. I, II, Maruzen, Tokyo, 1938, 1939; these Rev. 1, 86]. The most important of the n-dimensional theories under consideration are the ones by Kaluza, O. Klein, Einstein-Mayer, B. Hoffman, and Bergmann and O. Veblen. *V. Hlavatý.*

von Krbek, F. Anfangsgründe der allgemeinen Relativitätstheorie. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe 2, 23–36 (1953).

Expository paper.

Vaidya, P. C. 'Newtonian' time in general relativity. Nature 171, 260–261 (1953).

Taylor, N. W. A simplified form of the relativistic electromagnetic equations. Australian J. Sci. Research. Ser. A. 5, 423–429 (1952).

This theory is set in the curved space-time of general relativity, but the fourth coordinate is a pure imaginary. The basis is a skew-symmetric tensor $F_{\sigma\tau}$ with complex components, subjected to the condition of self-duality

$$\sqrt{g}\cdot g^{\mu\sigma}g^{\nu\tau}F_{\sigma\tau}=\tfrac{1}{2}\epsilon^{\mu\nu\sigma\tau}F_{\sigma\tau}$$

where ϵ is the permutation symbol. There are then only six real independent parts in $F_{\sigma\tau}$, and these are identified with the six components of the electric and magnetic vectors by formulae of the type $F_{12}=iE_3-H_3$ in a local cartesian system ($g_{\mu\nu}=\delta_{\mu\nu}$). The complete set of Maxwellian equations is contained in the complex tensor-density equation

$$\frac{\partial}{\partial x^\nu}(\sqrt{g}\cdot F^{\mu\nu})=J^\mu,$$

where J^μ represents the 4-current. There is a discussion of 4-potential, conservation of charge, ponderomotive force, and a variational principle. References are given to related work by other authors involving the bivectors $\mathbf{E}\pm i\mathbf{H}$ and quaternionic representations in the special theory of relativity. *J. L. Synge* (Dublin).

Infeld, L. The coordinate conditions and the equations of motion. Canadian J. Math. 5, 17–25 (1953).

Motion of a planet in the Schwarzschild field is considered and it is shown that, by a suitable change of angular variable, a coordinate system exists in which the orbit of the planet is a conic without perihelion displacement. In the two-body problem, it is argued that the equations of motion to the sixth approximation are independent of the choice of coordinate system but that this is not true for the eighth approximation. The equations of motion to the sixth approximation are thus not dependent on the choice of coordinate system provided that the author's approximation method be used. It is not, however, clear to this reviewer that the approximation procedure itself may not imply a choice of particular coordinate system. *G. C. McVittie.*

Ingraham, R. L. Spinor relativity. Nuovo Cimento (9) 10, 27–42 (1953).

The mathematical part of this paper quotes results from an as yet unpublished paper to establish relations between a spin connection and a connection of an underlying space, the space of conformal relativity and the curvatures derived from these connections. The physical part of the paper concerns the field equations derived from a variational principle, the Lagrangean of which involves a spinor field ψ, its covariant derivatives, and the scalar curvature of the conformal relativity space. One of the final remarks of this paper is: "This theory cannot be considered logically satisfying until the reason for the appearance of ψ is made clear." *A. H. Taub* (Urbana, Ill.).

Majumdar, Nandagopal. A note on the apparent disappearance of radiation in the theory of expanding universe. Bull. Calcutta Math. Soc. 44, 86–88 (1952).

A model universe with line-element

$$(1)\qquad ds^2=dt^2-e^{g(t)}(dx_1^2+dx_2^2+dx_3^2)$$

is considered, whose material content is partly matter and partly radiation. Assuming that a certain function of $g(t)$ and of the density of matter is approximately constant, it is possible to find an explicit expression for the density of radiation as a function of $g(t)$. This density tends to zero with the time. The author then considers the case when the initial density of radiation is zero, which he describes as that of a universe which has an "initial Einstein state." This cannot mean a universe with an initial "Einstein universe" state, for such a universe cannot be of type (1). He shows that the density of radiation attains a maximum value, before again tending to zero. *G. C. McVittie.*

Bandyopadhyay, G. New equation in the affine field laws. Physical Rev. (2) 89, 1161 (1953).

Roy, Louis. Expansion de l'Univers et champ cosmologique. C. R. Acad. Sci. Paris 236, 763–765 (1953).

Papapétrou, A. Les corpuscules à structure multipolaire en relativité restreinte. Prakt. Akad. Athēnōn 18 (1943), 40–50 (1950). (Greek summary)

In this theory the underlying space is the four-dimensional space of special relativity. The field of a particle is described by a tensor $\phi_{\alpha\beta}$ satisfying the equation $\Box\phi_{\alpha\beta}=0$, $\partial_\beta\phi^{\beta\alpha}=0$. Let $y^\alpha=y^\alpha(s)$ represent the world line C of the particle, $u^\alpha=\dot{y}^\alpha$ the velocity vector, and x^α a point for which $y^\alpha(s)$ is the retarded point of C. Then s and $n=u_\alpha(y^\alpha(s)-x^\alpha)$ are functions of x^α. The author gives a solution of the form $\phi_{\alpha\beta}=M_{\alpha\beta}n^{-1}+\partial_\gamma(M^\gamma{}_{\alpha\beta}n^{-1})$ where the $M_{\alpha\beta}$ and $M^\gamma{}_{\alpha\beta}$ are functions of s only. This means that the particle has bipolar character. Another solution corresponding to particles with a quadripolar character is discussed. *J. Haantjes.*

Papapétrou, A. La structure intérieure des corpuscules à constitution mono-bipolaire. Prakt. Akad. Athēnōn **18** (1943), 50–62 (1950). (Greek summary)

The author considers a particle moving uniformly about a fixed axis. The particle is supposed to have bipolar character. The material energy-tensor satisfies the equation $\partial_\beta T^{\alpha\beta}=0$. Starting from this equation and from some assumptions concerning the internal structure of the particle, it is shown that there exist two different kinds of particles of this type which have opposite angular momentum.

J. Haantjes (Leiden).

Papapétrou, A. Ondes gravifiques du corpuscule monobipolaire. Prakt. Akad. Athēnōn **18** (1943), 313–317 (1950). (Greek summary)

In this paper the author computes the flow of energy caused by gravitational waves produced by a particle of bipolar character. *J. Haantjes* (Leiden).

Papapétrou, A. La loi des moments dans un système quelconque de coordonnées. Prakt. Akad. Athēnōn **18** (1943), 317–323 (1950). (Greek summary)

In special relativity one has two conservation laws: (1) $\partial_\mu T^{\mu\lambda}=0$, $T^{\mu\lambda}$ being the material energy tensor, and the law (2) $\partial_\mu F^{\mu\lambda\kappa}=0$, $F^{\mu\lambda\kappa}=2T^{\mu[\lambda}x^{\kappa]}$, which is a consequence of the first one. In general relativity the first equation in the form $\nabla_\mu T^{\mu\lambda}=0$ has still a meaning, but, as the author shows, it seems impossible to find a generalization of (2).

J. Haantjes (Leiden).

Lalan, Victor. Sur une propriété caractéristique des transformations de Lorentz. Bull. Sci. Math. (2) **76**, 167–170 (1952).

A Lorentz transformation is defined as a linear transformation obtained from an ordinary Lorentz transformation ($x'=x, y'=y, z'=z \operatorname{ch}\phi+u \operatorname{sh}\phi, u'=z \operatorname{sh}\phi+u \operatorname{ch}\phi$) by a spatial rotation of the coordinate system. It is clear that the matrix of such a transformation is symmetrical. Let S denote a linear transformation $x^2+y^2+z^2-u^2$. The author proves: A transformation T is a Lorentz transformation if and only if it belongs to the connected set of transformations S containing the unit transformation. *J. Haantjes.*

Garnier, René. Sur une propriété caractéristique des transformations de Lorentz. Bull. Sci. Math. (2) **76**, 170–171 (1952).

The author gives another simple proof of the theorem of Lalan using a few properties of the characteristic roots of the symmetrical matrix belonging to a transformation S [see the preceding review]. *J. Haantjes* (Leiden).

Ingleton, A. W. The Lorentz transformation. Nature **171**, 618–619 (1953).

Clauser, Emilio. Trasformazioni nello spazio-tempo pseudoeuclideo che lasciano la metrica in forma statica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **13**, 116–120 (1952).

It is shown that in the space-time world of special relativity ($ds^2=dx_0^2-\sum dx_i^2$) a coordinate system can be introduced such that for this system $ds^2=g_0(x_i)(dx_0')^2-\sum(dx_i')^2$. The transformation $(x_0, x_i)\rightarrow(x_0', x_i')$ is computed. The new element is considered to describe the metric for an observer 0′ with a nonconstant velocity. *J. Haantjes.*

★**Graef Fernandez, Carlos. Birkhoff's theory of gravitation.** Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Diciembre, 1951, pp. 121–137. Centro de Cooperación Científica de la Unesco para América Latina, Montevideo, Uruguay, 1952. (Spanish)

This report contains a useful account of a theory that has hitherto gained little attention outside Latin America, namely the relativity theory originated by the late G. D. Birkhoff, and first fully propounded by him at the International Astrophysical Congress held in Mexico in 1942 [see Bol. Soc. Mat. Mexicana 1, no. 4–5, 1–23 (1944); these Rev. 6, 240]. The main body of the present report consists of a lecture by Graef Fernandez, in which he re-establishes the Birkhoff theory on the basis of seven postulates, none of which involves the concept of a perfect fluid such as was required in the original form of the theory. The first postulate is fundamental; it states that the frame of reference of gravitational phenomena is the flat space-time of Minkowski. (Birkhoff's theory, as the lecturer later emphasizes, is thus mathematically much simpler than the general-relativity theory of Einstein.) The second postulate is an adoption of the Einsteinian principle of equivalence; the third states that a gravitational field is completely characterized by a symmetric potential-tensor h_{ij} ($i, j=1, 2, 3, 4$). The fourth requires that, for a point-mass at rest in an inertial system, h_{ij} should be equal to the Newtonian potential times the Kronecker delta; the fifth that gravitational disturbances should be propagated with the velocity of light. The remaining two postulates lay down the additive nature of the gravitational potential for a single particle in motion and for a system of particles. The lecturer then outlines the way in which the Birkhoff theory may be developed on the basis of these postulates, giving references to literature where details of calculation or of argument have been omitted. Among the claims he puts forward for the theory, perhaps the most important are the following. (1) To a first approximation the Birkhoff theory gives the same results as general relativity for the three crucial experiments. (2) It provides for the interaction of gravitational and electromagnetic fields; that is (although the lecturer does not himself use the phrase), it is a unified theory. (3) It leads to a cosmology which turns out to be very similar to that of the late E. A. Milne.

The report concludes with a summary of the discussion that followed the lecture. Two main points emerged during the discussion, namely, that, although the theory has Minkowski space-time as its background, the equations of the paths of moving bodies have a fully covariantive form; and that, in the absence of an electromagnetic field, Birkhoff's theory is not equivalent to that of Einstein, even though the two theories then agree so far as observable effects are concerned. *H. S. Ruse* (Leeds).

MECHANICS

Kurovskiĭ, F. M. On the design of the motion schedule of a driven mechanism member. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 37–48 (1952). (Russian)

The paper provides the mathematical apparatus for design problems bearing on the kinematic performance of an oscillating driven member of a mechanism. The motion of the member is assumed symmetric about the midpoint of its time interval T of motion, and the author concentrates on the following laws of motion (a=acceleration): $a=a_0[1-(t/uT)^n]$, $n=m$ or $1/m$, $a=a_0[1-(1-2t/uT)^{2m}]$,

where $0 \leq u \leq \frac{1}{2}$, and m is an integer. The motion from $t=uT$ to $T-uT$ is assumed to be uniform. The absolute minimum peak acceleration $a_{0\infty}$ is obtained for $m=\infty$ (rectangular acceleration time diagram). For each of these types of motion the author derives expressions for a_0 and u in terms of T, the length of run S, and the ratio λ of maximum to average velocity; the expression for $\max \lambda$ at constant m; and the expression for $\min a_0 = a_{0m}$ at constant S, T, m. Applications contain problems in which $a_{0m}/a_{0\infty}$ has a prescribed lower bound, and a problem in which the timing of the peak velocity is prescribed. The paper is within the scope of a first college course in mechanics.

A. W. Wundheiler (Chicago, Ill.).

Matthieu, P. Über die Berechnung der Hypoidgetriebe. I. Ing.-Arch. **21**, 55–62 (1953).

About any two fixed skew lines as axes, two hyperboloids of revolution may be constructed, in an infinity of ways, so that the two hyperboloids have a straight line element in common. If these hyperboloids are rotated about their axes so that the linear speeds at their contacts are the same, the surfaces roll over each other with only lateral slipping. This is the basis of hyperboloidal (or hypoid) gearing. Teeth are added to the surfaces to ensure positive drive without relying on friction. As with spur gears and bevel gears, the tooth surface on one gear may, within limits, be taken as arbitrary. The shape of the mating tooth on the other gear is then the envelope of the positions of the selected tooth shape. The general polar coordinate equations of this desired envelope are derived in parametric form, suitable for computation, in terms of the equations of the selected tooth shape, the relative speeds of the gears and the angle between their fixed axes.

M. Goldberg (Washington, D. C.).

Fridlender, G. O. On the precession of a gyroscope under the action of an external moment. Akad. Nauk SSSR. Inženernyĭ Sbornik **12**, 229–233 (1952). (Russian)

In the usual investigations of the behavior of a gyroscope under the action of an external moment, the motion of the axle is examined, and the behavior of the kinetic moment vector and of the instantaneous angular velocity vector are disregarded. The paper is concerned with the behavior of these last two vectors in the case of a rapidly rotating flywheel.

E. Leimanis (Vancouver, B. C.).

Letov, A. M. On the theory of gyrosemicompasses. Akad. Nauk SSSR. Inženernyĭ Sbornik **13**, 123–130 (1952). (Russian)

A gyrosemicompass is an astatic gyroscope with three degrees of freedom whose axle is kept near the plane of the horizon. It is designed for stabilization of an arbitrarily chosen azimuth direction during a given time interval. Let $0\xi\eta\zeta$ be a right-hand orthogonal trihedral whose η-axis is oriented in the direction to be stabilized and whose ζ-axis points to the zenith of the observer. Let $0xyz$ be the Resal trihedral with the z-axis along the axle and pointing in the sense of positive values for the kinetic moment H of the gyroscope, and with the x-axis along the axis of the inner Cardan ring. Further, denote by M_x, M_y the moments of the exterior forces acting along the x-, y-axes respectively. Assume that in the initial position of the gyroscope at the instant $t=0$ the z-, η-axes, the y-, ζ-axes and the positive x-axis and the negative ξ-axis coincide. After elapse of a certain time interval a deviation of the gyroscope from the chosen direction will appear. This deviation can be characterized by two angles α and β, where β is the angle between the z-axis and its projection on the $\xi\eta$-plane, and α is the angle between this projection and the η-axis. The basic problem in the theory of gyrosemicompasses is to express the angles α, β as functions of the time.

The author considers two cases: (i) the free gyroscope ($M_x=M_y=0$); and (ii) the Anschütz azimuth gyroscope under the action of the moment of gravity

$$M_x = m_1 g l_1 \cos\beta = k_1 H \cos\beta$$

of a mass m_1 at the distance l_1 on the north end of the axle, and of the moment of gravity $M_y = m_2 g l_2 \sin\beta$ of a mass m_1 at the distance l_2 on the negative y-axis, the moments of the friction forces being neglected. In case (i) the author shows that any direction, determined by α_0, β_0, can be sufficiently accurately fixed by the axle of a gyroscope if the time t^* of observation is sufficiently small; the accuracy with which the chosen direction is indicated is determined by the deviations $\alpha(\alpha_0, \beta_0, t^*)$ and $\beta(\alpha_0, \beta_0, t^*)$. In case (ii) the following theorems are proved. Theorem 1. At any latitude φ and for arbitrary α_0, β_0 the axle of an azimuth gyroscope, free of friction forces, performs a regular conical precession with period T (depending on φ, diurnal rotation U of the Earth, α_0, β_0 and the parameter $k_2 = m_2 g l_2/H$) and well-defined maximum deviations for α and β. Theorem 2. For the given latitude φ the set of all azimuth directions for which the Anschütz gyroscope admits deviations not exceeding the given value α^* fills up a sector of angle α^* around the meridian of observation.

E. Leimanis (Vancouver, B. C.).

Slezkin, N. A. Generalization of Helmholtz's theorem on the resolution of the motion of a particle. Doklady Akad. Nauk SSSR (N.S.) **86**, 477–480 (1952). (Russian)

The author points out an analogy between motions of a physical system composed of discrete particles and Helmholtz's theorem for liquid motion. He sets up difference quotients analogous to components of vorticity and deformation rate for a tetrad of points, and studies changes that come about due to translations, rotations, and stretchings. He indicates that these operations can describe the transition from the liquid to the discrete system. The discussion is entirely theoretical, with no illustrations or examples.

R. E. Gaskell (Seattle, Wash.).

de Castro Brzezicki, A. Infinitesimal oscillations of dissipative systems. Gaceta Mat. (1) **4**, 11–14 (1952). (Spanish)

The author considers a dynamical system with two degrees of freedom, and subject to viscous friction, performing small oscillations about a configuration of equilibrium. Assuming that the effects of friction are small in a certain sense, he obtains formulae which describe the motion approximately. No discussion of the errors is given, and it is not apparent to the reviewer that the method used is in any way superior to other methods which are in common use.

L. A. MacColl (New York, N. Y.).

Šurova, K. E. Forming the variation of Poincaré's equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 123–124 (1953). (Russian)

Because of the singularly careless notations and about 20 bad misprints (including "holonomic" for "nonholonomic" in the statement of the scope which is not mentioned again), the reviewer believes that he should change the notations of the paper for the purpose of this review. The developments imply that a smooth nonholonomic system is considered, whose generalized velocities $\dot{x}^i$ ($i=1, \cdots, n$) are bound by

the equations $\dot{x}^i=c_\alpha{}^i\dot{q}^\alpha$ (the $\dot{q}$ are not time derivatives), $\alpha=1, \cdots, k$. Let $\partial/\partial x^i=\partial_i$ (the x themselves are independent), $X_\alpha=\xi_\alpha{}^i\partial_i$, and

$$X_\alpha X_\beta-X_\beta X_\alpha=c_{\alpha\beta}{}^\gamma X_\gamma \quad (\alpha, \beta, \gamma=1, \cdots, k).$$

Let L be the Lagrangian expressing $T-V$ in terms of x, $\dot{q}$, and the time t; let $p_\alpha=\partial L/\partial\dot{q}_\alpha$; and let H be the Hamiltonian expressing $p_\alpha\dot{q}^\alpha-L$ in terms of x, p, t. From the equations $\dot{p}_\alpha=c_{\alpha\beta}{}^\gamma\dot{q}^\beta p_\gamma-X_\alpha L$, commonly attributed to Boltzman, Volterra, or Woronetz (the authoress credits them to Poincaré who derived them for the case of constant c coefficients, the only one she seems to consider) the following equations are easily obtained for the variations of x and p ($\delta x^i=\xi_\alpha{}^i\delta q^\alpha$, $\partial^\alpha=\partial/\partial p_\alpha$):

$$\delta q^\beta X_\beta\partial^\alpha H=\partial^\alpha(\delta q^\beta X_\beta H+\delta p_\beta\partial^\beta H),$$
$$d\delta p_\alpha/dt=c_{\alpha\beta}{}^\gamma p_\gamma\delta q^\epsilon X_\epsilon\partial^\beta H+c_{\alpha\beta}{}^\gamma\delta p_\gamma\partial^\beta H -X_\alpha(\delta q^\beta X_\beta H+\delta p_\beta\partial^\beta H).$$

These equations hold for constant c's only. [For the general case, including rheonomic constraints, see the reviewer's paper in Prace Mat.-Fiz. 38, 129–146 (1931).]

A. W. Wundheiler (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

✱Sedov, L. I. Metody podobiya i razmernosti v mehanike. [Similarity and dimensional methods in mechanics.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, Leningrad, 1951. 193 pp. 10.20 rubles.

This is the second edition of a book published in 1944, but not previously reviewed in Math. Reviews. Intended primarily as an expository survey, the book is notable for the wealth of examples analyzed. Many of the discussions are based on previously published research of the author. Its scope is comparable with that of Chaps. III–IV of the reviewer's "Hydrodynamics: A study in logic, fact and similitude" [Princeton, 1950; these Rev. 12, 365] except that its inclusion of specific experimental data resembles more the exposition of H. L. Langhaar, "Dimensional analysis and theory of models" [Wiley, New York, 1951; these Rev. 12, 580]. Also, the relation of dimensional analysis to group theory is not treated.

Chapter I (30 pp.) is a conventional discussion of dimensional analysis, leading up to the Π-theorem. Chapter II (67 pp.) deals with the influence of Reynolds number on pipe resistance (Stanton-Pannell) and on drag, with the influence of Reynolds and Mach number on compressible flow, with the Strouhal number, with the influence of the Froude number on planing and impact of seaplane hulls, with Wagner's discussion of similitude ("automodel" solutions) for cone and wedge impact, and with similitude for gravity waves. Chapter III (50 pp.) deals with viscosity and turbulence. First, he treats the diffusion of vortices, other exact solutions of the Navier-Stokes equations (which seem related to those of Hamel), Blasius' solution of the boundary layer equations. Then homogeneous turbulence is analyzed following the work of Millionščikov [Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 1941, 433–446; these Rev. 4, 121] and Kolmogoroff; this is followed by a brief discussion of turbulent flow in pipes, using "mixing length" concepts. Chapter IV (45 pp.), which was apparently added in the second edition, deals with the unsteady motion of gases. Emphasis is placed on the self-similar ("automodel") solutions of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, 91–93 (1945); these Rev. 7, 140; Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 241–250 (1946)]. The piston problem, spherical blast waves, detonation and burning, and atomic explosions are discussed, the last following G. I. Taylor [Proc. Roy. Soc. London. Ser. A. 201, 175–186 (1950)].

G. Birkhoff (Cambridge, Mass.).

Ertel, Hans. Über die physikalische Bedeutung von Funktionen, welche in der Clebsch-Transformation der hydrodynamischen Gleichungen auftreten. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Naturwiss. 1952, no. 3, 19 pp. (1952).

There are several known kinematic characterizations of classical hydrodynamics: Kelvin's circulation theorem, Helmholtz's vorticity theorems, Cauchy's vorticity formula, the Clebsch transformation, the Weber transformation, etc. In this review sufficient stipulations of smoothness will be left for the reader, and all results are local. The Stuart-Lamb extension (1900) [Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, see §167] of Clebsch's transformation may be put as follows. Let f and g satisfy

$$\text{(1)} \qquad \operatorname{grad} \dot{f}\times\operatorname{grad} g=\operatorname{grad} \dot{g}\times\operatorname{grad} f;$$

then, for any h, the velocity field

$$\text{(2)} \qquad \dot{\mathbf{x}}=\operatorname{grad} h+f\operatorname{grad} g$$

is circulation-preserving. If, conversely, f, g, h are Monge potentials (not unique) of a circulation-preserving motion (2), whose acceleration potential is, say, Φ, then not only is (1) satisfied but also if

$$\text{(3)} \qquad \lambda=\Phi+\tfrac{1}{2}\dot{x}^2+\frac{\partial h}{\partial t}+f\frac{\partial g}{\partial t},$$

it follows that $\lambda=\lambda(f, g, t)$ and that λ satisfies the canonical system

$$\text{(4)} \qquad \frac{\partial\lambda}{\partial g}=\dot{f}, \quad \frac{\partial\lambda}{\partial f}=-\dot{g}.$$

However, if Φ is the acceleration potential for a given circulation-preserving motion $\dot{\mathbf{x}}$, that f, g, λ, h satisfy (1), (3), and (4) is not sufficient for (2) to hold. The original result of Clebsch (1859) is a statement that it is always possible, in the circulation-preserving case, to choose f, g, and h so that $f=\text{const.}$, $g=\text{const.}$ are congruences of material vortex-surfaces, whence follows $\dot{f}=0$, $\dot{g}=0$, and thus by (4) follows $\lambda=0$. The author's stated objective is to give, for the case of steady motion, a physical interpretation for "the" functions occurring in (2). In fact, however, he does not attack the general problem, being content to construct a special choice of f, g, h, different from that of Clebsch.

Weber's transformation (1868) [e.g., Lamb, op. cit., §15] can be put as follows. Write

$$\text{(5)} \qquad W=\int_0^T [\tfrac{1}{2}\dot{x}^2-\Phi]dt.$$

Then at time T

$$\text{(6)} \qquad \dot{x}_i=\dot{X}_\alpha X^\alpha{}_{,i}+W_{,i},$$

where X^α are the material coordinates, $X^\alpha{}_{,i}=\partial X^\alpha/\partial x^i$, and $\dot{X}_\alpha$ is the velocity vector at time $T=0$. The author in effect ingeniously modifies (5) by putting

$$\text{(7)} \qquad W=\int_\vartheta^T [\tfrac{1}{2}\dot{x}^2-\Phi]dt,$$

where $\vartheta=\vartheta(X^\alpha)$ is an initial time which may vary arbitrarily from one particle X^α to another. Then the analogue

of the usual derivation of (6) from (5) yields at once

$$\dot{x}_i x^i{}_{,\alpha}\Big|_{t=\vartheta}^{t=T} = W_{,\alpha} + H\Theta_{,\alpha}, \tag{8}$$

where $\Theta = t - \vartheta$ and where $H = \frac{1}{2}\dot{x}^2 + \Phi =$ const. on each streamline. Hence, if $\bar{\mathbf{x}} = \mathbf{x}(\mathbf{X}, \vartheta)$, while $\bar{\dot{\mathbf{x}}} = \dot{\mathbf{x}}|_{t=\vartheta}$, we have

$$\dot{x}_i = \bar{\dot{x}}_j \bar{x}^j{}_{,i} + W_{,i} + H\Theta_{,i}. \tag{9}$$

If the first term on the right-hand side vanishes, (9) reduces to the form (2).

Now consider only those $\mathbf{X}$ which lie on a certain surface. Then the $\mathbf{x}$ lie on a surface. For fixed i, $\bar{x}^j{}_{,i}$ is a vector tangent to this surface. If the velocity $\bar{\dot{\mathbf{x}}}$ is normal to this surface, $\bar{\dot{x}}_j \bar{x}^j{}_{,i} = 0$. Hence from (9) follows the author's beautiful result: In a circulation-preserving steady motion such that there exists one surface F which is cut orthogonally by every streamline, it is possible to choose the Monge potentials f, g, h in (1) so that $h = W =$ "Hamilton's action function" (as modified by the author), $f = H =$ the total energy per unit mass ("Bernoulli function"), $g = \Theta =$ time since the particle now at $\mathbf{x}$ crossed the surface F. (This choice of potentials satisfies the Hamiltonian equations (4) by $f = 0$, $\lambda = \lambda(f) = f$, $g = -1$, while (2) is trivially satisfied.)

The author remarks also that if we put $S \equiv W + H\Theta$, then $\dot{\mathbf{x}} = \operatorname{grad} S - \Theta \operatorname{grad} H$ gives another set of Monge potentials with simple dynamical interpretation. He also discusses the connection between these results and two earlier vorticity theorems [Ertel, Miscellanea Academica Berolinensia, vol. I, pp. 62–68, Akademie-Verlag, Berlin, 1950; these Rev. **14**, 408; Ertel and Rossby, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1949**, no. 1; Geofis. Pura Appl. **14**, 189–193 (1949); these Rev. **12**, 58, 367]. He promises an extension to unsteady flows in a later work. *C. Truesdell.*

v. Gorup, Guntram. Eine neue Methode zur Berechnung der Strömungsfunktionen bei zeitlich veränderlicher Kontur. Z. Angew. Math. Mech. **32**, 371–378 (1952). (English, French and Russian summaries)

In this paper the author considers the two-dimensional problem of a body having a boundary which varies with the time, in motion in an inviscid fluid. From the usual boundary condition that the relative velocity between the fluid and the body at the boundary must be tangential, he derives a functional equation for the stream function. Assuming that the mapping function for the boundary is meromorphic and given in a closed form, this functional equation together with the regularity conditions in the space outside the boundary, are sufficient to determine the stream function in closed form. The Joukowski profile having a variable thickness and camber is treated as an example. *R. M. Morris* (Cardiff).

Szebehely, V. G. Generalization of the dimensionless frequency parameter in unsteady flows. The David W. Taylor Model Basin, Washington, D. C., Rep. **833**. iii+22 pp. 1952

The acceleration vector $\mathbf{a}$ of fluid motion can be regarded as the sum of two vectors, $\mathbf{a}_l = \partial\mathbf{v}/\partial t$, where $\mathbf{v}$ is the velocity vector, and $\mathbf{a}_c$ the convective acceleration. The author defines a generalized Strouhal number $S = |\mathbf{a}_l|/|\mathbf{a}_c|$ which he takes to be a quantitative measure of unsteadiness. He points out the importance of accelerationless flows for which $\mathbf{a} = 0$, $\mathbf{a}_l \neq 0$, the value of S being then unity. In general, S is a function of position $\mathbf{r}$ and time t. In the case of separable or d'Alembert flows, for which $\mathbf{v} = T(t)\mathbf{u}(\mathbf{r})$, S is also separable.

Three examples are discussed in detail: an unsteady jet, the decay of vorticity, and the inertia method of measuring ship resistance. An unsteady jet striking a plate has maximum unsteadiness on the plate and minimum unsteadiness on the axis of the jet. The two types of vorticity dissipation studied are fundamentally different. A circular eddy is represented by a nonseparable velocity distribution and a set of "rectangular" vortices by a d'Alembert flow. The velocity dies out and the unsteadiness increases with time in both cases. The assumptions made in the inertia method of measuring ship resistance are equivalent to assuming time-independent unsteadiness. The ideas embodied in this paper seem to the reviewer to open up a promising field of research. *L. M. Milne-Thomson* (Greenwich).

Garabedian, P. R., Lewy, H., and Schiffer, M. Axially symmetric cavitational flow. Ann. of Math. (2) **56**, 560–602 (1952).

This paper contains the first general existence proof in the theory of axially symmetric flows with free boundaries. The authors fix their attention on uniform flows past a finite cavity, the cavity being a body of revolution generated by an arc in the meridian plane, its image in a plane perpendicular to the axis of symmetry, and a free streamline joining the endpoints of the arcs (Riabouchinsky cavity). Following the method of Garabedian and Spencer [J. Rational Mech. Anal. **1**, 359–409 (1952); these Rev. **14**, 102] for plane cavitational flows, they deduce the existence of such axisymmetric cavity flows as the solution of a free boundary variational problem. This problem seeks among all bodies of revolution generated by curves containing two symmetrically disposed fixed monotonic arcs in common the body achieving the minimum value of $M - (\lambda - 1)V$, where M is the virtual mass of the body in the flow direction, V is the volume enclosed by it, and λ is a constant. If the velocity of the incident flow is taken to be unity, then the flow past the extremal body is a cavity flow in which λ is the square of the constant flow speed on the free streamline. In the existence proof symmetrization provides a tractable minimizing sequence having a rectifiable free boundary as limit, and the method of interior variations shows that a weakened form of the constant flow speed condition is satisfied on the free boundary. The chief element of difference from the plane case is in the proof of analyticity of the free boundary which at the same time provides the constant flow speed condition in full strength. This proof is based on the analytic continuation of the equation for the stream function into the complex domain of the independent variables. A by-product of this analysis is an interesting formula for the stream function of an axially symmetric flow with prescribed free boundary by means of which axially symmetric free surface flows can be generated explicitly. The existence of the infinite Helmholtz cavity is obtained in the limit by letting the symmetric image of the fixed arc in the finite cavity move to infinity downstream. Included in the paper are also theorems of uniqueness and continuous dependence on the parameter λ. This paper marks an important advance in extending the results of the plane free-boundary problem to spatial flows. *D. Gilbarg* (Bloomington, Ind.).

Keller, Joseph B., and Goldstein, Edward. Water wave reflection due to surface tension and floating ice. Trans. Amer. Geophys. Union **34**, 43–48 (1953).

Le problème considéré est celui de la propagation d'une houle sinusoïdale qui rencontre une zone de la surface libre,

constituée par un demi-plan, qui possède une tension superficielle ou qui est recouverte d'une couche matérielle mince. Ce problème qui a déjà fait l'objet des études de Peters [Comm. Pure Appl. Math. **3**, 319–354 (1950); ces Rev. **12**, 869] et de Weitz et Keller [ibid. **3**, 305–318 (1950); ces Rev. **12**, 762] est traité ici dans le cas où la profondeur est faible. Les auteurs montrent qu'il existe généralement une onde réfléchie et deux ondes transmises, dont l'une est amortie (cette dernière existant seulement dans le cas de la tension superficielle). La direction de l'onde incidente étant supposée faire un certain angle avec la ligne de séparation des deux régions de la surface libre, les auteurs introduisent les angles et les coefficients de réflexion et de réfraction. Ces angles et ces coefficients sont explicités en fonction des caractéristiques de l'onde incidente, de la valeur de la tension superficielle, et de la densité de la couche matérielle flottante. *R. Gerber* (Toulon).

***Ursell, F. Discrete and continuous spectra in the theory of gravity waves.** Gravity Waves, pp. 1–5. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

The content of this paper has been expounded in greater detail in Proc. Roy. Soc. London. Ser. A. **214**, 79–97 (1952); these Rev. **14**, 326. *J. V. Wehausen* (Providence, R. I.).

***Davies, T. V. Symmetrical, finite amplitude gravity waves.** Gravity Waves, pp. 55–60. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

The results of this paper have already appeared in a more detailed exposition [Davies, Quart. Appl. Math. **10**, 57–67 (1952); Proc. Roy. Soc. London. Ser. A. **208**, 475–486 (1951); Packham, ibid. **213**, 238–249 (1952); these Rev. **13**, 698, 396; **14**, 216]. *J. V. Wehausen* (Providence, R. I.).

***Pohle, Frederick V. Motion of water due to breaking of a dam, and related problems.** Gravity Waves, pp. 47–53. National Bureau of Standards Circular 521, U. S. Government Printing Office, Washington, D. C., 1952. $1.75.

The author takes the two-dimensional equations of motion for a perfect fluid in the Lagrangian representation. Assuming that the displacements and pressure are analytic functions of the time t, he expands them in powers of t and derives formulas for relating the coefficients of t^n to the preceding coefficients. This gives a convenient method for handling free-boundary problems, which the author illustrates with the case of the flow immediately following the breaking of a dam and briefer reference to two similar problems. *J. V. Wehausen* (Providence, R. I.).

Moiseev, N. N. The problem of the motion of a rigid body containing a liquid mass having a free surface. Mat. Sbornik N.S. **32**(74), 61–96 (1953). (Russian)

The author considers the problem of the title under the following assumptions: ideal incompressible fluid; potential motion; extraneous forces all in one direction and functions of time only; linearized boundary conditions. In §1 the author treats oscillations in a fixed basin by expanding the velocity potential $\varphi(x, y, z, t)$ in series of functions orthogonal on the curve of intersection of the basin walls and the undisturbed free surface. The coefficients are functions of t alone and infinite sets of differential equations for these are obtained. In §2 the equations are extended to the case of a movable basin. In §§3 and 4 the equations of motion for the basin with liquid are derived; an energy integral is also derived for the case of conservative forces. §5 takes up the case of small oscillations of the basin when the external forces have a potential. Positive definiteness of a certain quadratic form is necessary and sufficient for stability, an analogue of the situation in dynamics of rigid bodies. In §6 the case of small oscillations with one degree of freedom is discussed in some detail; the effect of dissipative forces is also considered. In §7 most of the developed theory is applied to the special case of a basin in the shape of a rectangular parallelepiped. *J. V. Wehausen.*

Moiseev, N. N. Motion of a rigid body having a hollow partly filled by an ideal liquid. Doklady Akad. Nauk SSSR (N.S.) **85**, 719–722 (1952). (Russian)

The equations of motion of the system, basin+liquid, are derived in a different form from that in §§3 and 4 of the paper reviewed above. Application is made to small oscillations of one degree of freedom of a rectangular parallelepiped with a linear restoring force (as in the last part of §7 of the paper reviewed above). *J. V. Wehausen.*

Moiseev, N. N. On oscillations of a heavy ideal incompressible liquid in a basin. Doklady Akad. Nauk SSSR (N.S.) **85**, 963–965 (1952). (Russian)

This note is essentially a summary of the results in §1 of the paper reviewed second above. *J. V. Wehausen.*

Moiseev, N. N. On two pendulums filled with liquid. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 671–678 (1952). (Russian)

The author applies the methods developed in the paper reviewed third above to small oscillations of a suspended open basin containing liquid, two types of suspension being considered. In the first type the basin is suspended by a pair of parallel rods attached at each end of the basin and hinged so that the basin swings parallel to itself. In the second type the basin is rigidly attached to the supporting rod. The equations of motion for each type are derived and the stability of the motion discussed. The equilibrium position of the first is always stable whereas that of the second may be unstable. The equations of motion are solved and the location of the characteristic frequencies discussed.

J. V. Wehausen (Providence, R. I.).

Haskind, M. D. Two papers on the hydrodynamic theory of heaving and pitching of a ship. The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No. 1-12, 60 pp. (1953).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. **10**, 33–66 (1946); Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1946**, 23–34; these Rev. **11**, 228; **8**, 71.

Gomes, A. Pereira. Détermination d'un "convergent" ayant un corps à l'ouverture de sortie. Portugaliae Math. **12**, 49–56 (1953).

The method of H. S. Tsien for constructing a convergent channel flow with axial symmetry and no adverse pressure gradients along the wall is extended to the case of a channel with an axially symmetric obstruction in the exit. This extension is accomplished by superimposing a doublet on the channel flow. Tables are given for a particular case.

P. W. Ketchum (Urbana, Ill.).

Nickel, K. Über spezielle Tragflügelsysteme. Ing.-Arch. **20**, 363–376 (1952).

This paper is concerned with two classes of wing problems that are closely related mathematically; namely, (i) flow

about a number n of lifting-line wings flying beside one another in the same horizontal plane, treated according to the familiar Prandtl theory, and (ii) a number n of thin airfoils flying one behind another, treated according to the thin-airfoil approximation of Birnbaum et al. In both cases, the mathematical problem takes the form

$$(1)\qquad u(\zeta)-iv(\zeta)=\frac{i}{2\pi}\sum\int_C\frac{\gamma(z)}{\zeta-z}dz$$

where z and ζ are complex variables and the summation indicates that the contour integrals are to be taken over all those intervals of the axis of reals (wings) where the vortex strength γ differs from zero. The problem of interest is to determine γ over these intervals when v is prescribed on the same intervals.

For a single wing, the inversion of (1) is well known; it is attributed here to Betz [Göttinger Dissertation, Oldenbourg, München, 1919]. For the case of n intervals it has been generalized by the author in an earlier paper [Math. Z. 54, 81–96 (1951); these Rev. 13, 43]. Here the result is applied first to case (i), where the total circulation about each wing must vanish. The minimum-drag problem, where v has the same constant value on each interval, is discussed and is worked out in detail for $n=2$. In case (ii) the Kutta-Joukowski condition $\gamma=0$ is imposed at each trailing edge. The total lift and moment are calculated, and some general results concerning the properties of a multiple-slotted thin airfoil are found.

A mathematical appendix includes proofs of certain lemmas previously used and a table giving the values of the following sums of definite integrals:

$$\sum_{\nu=1}^{n}\int_{a_\nu}^{b_\nu}R^{\pm1}\xi^k d\xi \quad \text{for}\quad k=0, 1, 2,$$

and

$$\sum_{\nu=1}^{n}\mathrm{P}\int_{a_\nu}^{b_\nu}R^{\pm1}\frac{\xi^k d\xi}{x-\xi} \quad \text{for}\quad k=0, 1, 2, 3,$$

where

$$R=\left(-\prod_{\lambda=1}^{n}\frac{a_\lambda-\xi}{b_\lambda-\xi}\right)^{1/2},$$

$$a_1<b_1<a_2<b_2<\cdots<a_n<b_n,\quad\text{and}\quad a_1<x<b_n,$$

and P denotes Cauchy's principal value. *W. R. Sears.*

Jaeckel, K. Vereinfachte Herleitung der Weissingerschen Zirkulationsgleichung für den schiebenden Tragflügel. Z. Angew. Math. Mech. 33, 65–66 (1953).

The downwash $w(y)$ at a yawed lifting line is expressed as the sum of several terms, of which all are finite except one, $w_3(y)$, which exhibits the well-known logarithmic singularity. The author shows that this singular term can be reproduced as the downwash at the trailing edge due to a certain (integrable) circulation distribution $\gamma_3(x, y)$ along the chord at y. Consequently, he calculates the total circulation $\Gamma(y)$ in the Prandtl fashion, using the finite downwash terms to express the induced angle, and adds the integral of $\gamma_3(x, y)$, say $\Gamma_3(y)$ to the result. The final expression for $\Gamma(y)$ then resembles Weissinger's [Jbuch. Deutsch. Luftfahrtforschung **1940**, I145–I181; see also NACA Tech. Memo. no. **1120** (1947); Math. Nachr. 2, 45–106 (1949); these Rev. 8, 542; 11, 64]. *W. R. Sears.*

Lambourne, N. C. On the conditions under which energy can be extracted from an air stream by an oscillating aerofoil. Aeronaut. Quart. 4, 54–68 (1952).

***Sedney, R., Charnes, A., and Saibel, E. The Reynolds lubrication equation with smooth outflow.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 875–881. The American Society of Mechanical Engineers, New York, N. Y., 1952.

A generalization of the boundary conditions in the Reynolds hydrodynamical theory of lubrication is obtained as natural boundary conditions arising from a variational principle. *From the authors' summary.*

Sowerby, L. The couple on a rotating spheroid in a slow stream. Proc. Cambridge Philos. Soc. 49, 327–332 (1953).

In the case of an oblate spheroid rotating uniformly about its axis of symmetry in a steady stream of viscous fluid moving parallel to this axis, the author shows that the transverse velocity component can be obtained in terms of spheroidal wave functions if the Oseen approximation is permitted. With this velocity component, the couple required to maintain the steady rotation is determined. When the mainstream velocity is put equal to zero, the result agrees with that of Jeffery [Proc. London Math. Soc. (2) **14**, 327–338 (1915)]. In the limiting case of vanishing thickness, the spheroid becomes a disc and the corresponding couple is calculated numerically in a range of Reynolds numbers. *Y. H. Kuo* (Ithaca, N. Y.).

Oudart, Ad. Mise en régime de la couche limite de la plaque plane dans l'impulsion brusque à partir du repos. Recherche Aéronautique no. 31, 7–12 (1953).

For a plate set impulsively into motion with a constant velocity U_∞, the author wishes to know the time necessary for the steady state to be established. This problem is solved by the momentum-integral method. It is shown that for laminar flow this time required is $2.86x/U_\infty$, x being the distance measured from the leading edge in the direction of the motion. When the flow is turbulent, the numerical factor is found to be 1.285 for the 1/7th power velocity profile. (The same problem but from a broader point of view was attacked previously by Stewartson [Quart. J. Mech. Appl. Math. 4, 182–198 (1951); these Rev. 13, 82].) *Y. H. Kuo* (Ithaca, N. Y.).

Thwaites, B. On the flow past a flat plate with uniform suction. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2481** (**9391**), 11 pp. (1952).

By a method due to the author [same Rep. and Memoranda no. **2243** (1946); these Rev. 10, 76], the velocity profile in the boundary layer is expressed by $y/\vartheta=f(t)$ and $t=u/U$ where y, ϑ, u are respectively the coordinate normal to the body, the momentum thickness and the velocity parallel to the boundary; U is the value of u at the edge of the boundary layer. In the case of the porous flat plate, if the suction is constant and if the velocity profile is a superposition of the Blasius profile $f(t)$ and the asymptotic profile $g(t)$ for constant suction, i.e., $(1-K)f(t)+Kg(t)$, where K is a function of x (measured along the plate) and lies between 0 and 1, then the momentum integral yields an explicit solution. The results compare favorably with those of Schlichting. The question of an exact solution is briefly mentioned. *Y. H. Kuo* (Ithaca, N. Y.).

Illingworth, C. R. The laminar boundary layer associated with the retarded flow of a compressible fluid. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2590** (9886) 23 pp. (1953).

In this report (written in 1946), the author first proves that for a compressible boundary-layer flow a similarity parameter exists only if the main stream is uniform. In the second part of the report, the Kármán-Pohlhausen method is applied for both constant-pressure and constant-pressure-gradient cases. For different velocity profiles, numerical data for all the cases considered are included.

Y. H. Kuo (Ithaca, N. Y.).

Tatsumi, Tomomasa. Note on discrepancies between two theories on the stability of plane Poiseuille flow. J. Phys. Soc. Japan **7**, 619–624 (1952).

The author compares the work of Pekeris [Physical Rev. (2) **74**, 191–199 (1948); these Rev. **10**, 270] and Lin [Quart. Appl. Math. **3**, 117–142, 218–234 (1945); 277–301 (1946); these Rev. **7**, 225, 226, 346] on the stability of plane Poiseuille flow. Both authors use the classical method of small wavy disturbances but differ in the mathematical methods used. It is pointed out that Pekeris' formula for the characteristic value is of limited application and hence he cannot find the instability found by Lin, with whose work the author concurs. [See also L. H. Thomas, Physical Rev. (2) **86**, 812–813 (1952); these Rev. **14**, 697]. The author also takes issue with Pekeris on the class of solutions with dimensionless wave speed close to unity. *C. C. Lin.*

Lin, C. C. On the stability of the laminar mixing region between two parallel streams in a gas. NACA Tech. Note no. **2887**, 50 pp. (1953).

C'est une étude de la stabilité de la couche limite séparant deux écoulements gazeux uniformes. Il est d'abord établi que si la vitesse relative des deux écoulements est supérieure à la somme de leur vitesse du son il ne peut y avoir propagation subsonique d'une perturbation. La couche limite doit alors être stable. Les petites oscillations sont ensuite étudiées en négligeant la viscosité et la conductibilité thermique. Il est montré qu'il peut exister une et une seule perturbation périodique de longueur d'onde finie, moyennant certaines conditions qui sont explicitées. Les petites oscillations sont alors amplifiées ou amorties, suivant que leur longueur d'onde est plus grande ou plus petite que celle de cette perturbation périodique. Pour terminer, des développements de la théorie des solutions asymptotiques font voir, en particulier, que pour des nombres de Reynolds élevé les oscillations amorties dans un courant uniforme ont une structure semblable à celle du champ du rotationnel dans un écoulement turbulent. *R. Gerber* (Toulon).

Lin, C. C. A simplified formulation of the similarity concepts in isotropic turbulence. J. Aeronaut. Sci. **20**, 286 (1953).

Uberoi, Mahinder S. Quadruple velocity correlations and pressure fluctuations in isotropic turbulence. J. Aeronaut. Sci. **20**, 197–204 (1953).

Calcul exact, en turbulence homogène, des corrélations de pression en deux points, en fonction des corrélations quadruples de vitesse en ces deux points. Simplification du résultat dans le cas de la turbulence isotrope, puis en faisant l'hypothèse que les corrélations quadruples de vitesse sont liées aux corrélations doubles par la même relation que dans le cas de variables gaussiennes. Application au carré moyen du gradient de pression. Relation avec l'écart type de la vitesse en variable de Lagrange.

Quelques vérifications expérimentales: construction directe des courbes de corrélation quadruple. Comparison avec les courbes calculées, grâce à l'hypothèse ci-dessus, à partir des valeurs expérimentales des corrélations doubles. Application au tracé des courbes de corrélation des pressions. Discussion des résultats. *J. Bass* (Chaville).

Krzywoblocki, M. Z. On the fundamentals of locally isotropic turbulence in magneto-hydrodynamics of a compressible medium. Acta Physica Austriaca **6**, 250–256 (1953).

Starting from the equations of isotropic turbulence in a compressible fluid which the author has recently derived [same Acta **6**, 157–166 (1952); these Rev. **14**, 596] he considers the case when there is a constant dissipation of energy (due to electrical conductivity, thermal conductivity, and viscosity) and the various correlations one may write down between the relative velocities and the relative magnetic field strengths at two different points depend only on the distance between them and are independent of time, i.e., when conditions appropriate to locally isotropic turbulence prevail. *S. Chandrasekhar* (Williams Bay, Wis.).

Chandrasekhar, S. On the inhibition of convection by a magnetic field. Philos. Mag. (7) **43**, 501–532 (1952).

The stability of a layer of fluid heated from below is investigated in the case when the fluid is considered an electric conductor and an external magnetic field is impressed on the fluid, especially in the direction of the gravitational force. The problem is first investigated on the assumption that the "principle of exchange of stabilities" is valid. The relevent disturbances are of the nature of a cellular convection. The critical Rayleigh numbers are calculated by means of variational principles for three sets of boundary conditions. The inhibiting effect of the magnetic field is the greater, the stronger the field and the larger the electric conductivity. It is shown that under astrophysical conditions, instability may arise either as the familiar cellular convection, or as oscillation of increasing amplitude.

C. C. Lin (Cambridge, Mass.).

Chandrasekhar, S. The stability of viscous flow between rotating cylinders in the presence of a magnetic field. Proc. Roy. Soc. London. Ser. A. **216**, 293–309 (1953).

This paper deals with the stability of viscous flow between cylinders rotating in the same direction and in the presence of a magnetic field along the axis. Critical Taylor numbers are calculated by variational methods. It is found that the inhibiting effect of the magnetic field is the greater, the stronger the field and the larger the electric conductivity. This effect is similar to that in the case of thermal instability [see the preceding review], and is more pronounced in the present case. *C. C. Lin* (Cambridge, Mass.).

Lehnert, Bo. On the behaviour of an electrically conductive liquid in a magnetic field. Ark. Fys. **5**, 69–90 (1952).

In this paper the following three simple cases of steady viscous flow of an electrically conducting fluid in a magnetic field are considered: (1) The flow between rotating cylinders with an impressed magnetic field along the axis. In this case the character of the flow is unchanged by the presence of the magnetic field. (2) The flow between two parallel planes separated by a distance, $2d$, and sheared with a relative velocity, $2v_0$. The impressed magnetic field, H, is in the

transverse direction. In the absence of the field, the velocity profile is linear between the planes. In the presence of the field this becomes $v=v_0 \sinh kz/\sinh kd$ where $k=\mu H(\sigma/\rho\nu)^{\frac{1}{2}}$ where σ, ν, and μ are the coefficients of electrical conductivity, kinematic viscosity, and magnetic permeability, respectively, and z is measured from the midpoint between the two planes. (3) The flow between two parallel planes under a constant pressure gradient and when the impressed magnetic field is in the transverse direction. This is the case first considered by Hartmann [Hartmann, Danske Vid. Selsk. Math.-Fys. Medd. **15**, no. 6 (1937); Hartmann and Lazarus, ibid. **15**, no. 7 (1937)]. In the absence of a field the velocity profile is parabolic; in the presence of the field this becomes $v=v_1(\cosh kd-\cosh kz)/\sinh kd$ where v_1 is the velocity at the midpoint and k has the same meaning as in case (2). The principal characteristic of the solutions in cases (2) and (3) is that when the impressed field becomes sufficiently strong, the flow acquires the character of a boundary layer flow with a thickness of the layer of the order of $1/k$.

Some space is devoted to defining parameters which may characterize the stability of viscous flow in the presence of a magnetic field. In addition to the Reynolds number $R(=lv/\nu)$ these may be taken to be $Q=\mu^2H^2\sigma l^2/\rho\nu$ and $S=R(lv\mu\sigma)$ and combinations.

In the last section some brief comments on the stability of viscous flow between rotating cylinders are made and some new experimental data are presented. [The theoretical solution of this last problem has since been given by the reviewer in the paper reviewed above.]

S. Chandrasekhar (Williams Bay, Wis.).

Lundquist, Stig. Studies in magneto-hydrodynamics. Ark. Fys. **5**, 297–347 (1952).

This paper gives a general account of the growing literature on magneto-hydrodynamics. The following topics among others are considered: In Section II the basic equations of the problem including conductivity and viscosity are written down. In Section III it is shown that for an incompressible fluid the quantities $\mathbf{v}\pm\mathbf{H}/(4\pi\rho)^{\frac{1}{2}}$ (where $\mathbf{v}$ denotes the velocity and $\mathbf{H}$ the intensity of the magnetic field) satisfy equations which are entirely symmetrical. In Section IV the well known analogy between the magnetic field in an infinitely conducting medium and the vorticity in an inviscid incompressible fluid is elaborated. In Section V some special solutions are considered for the stationary case when the term in curl $\mathbf{H}\times\mathbf{H}$ in the equations of motion is balanced by the pressure gradient. In particular, the problem treated earlier by the writer [Physical Rev. (2) **83**, 307–311 (1951); these Rev. **13**, 189] of the stability of twisted fields is reconsidered under slightly more general conditions. In Section VI the manner in which a given initial magnetic field gets modified by a prevalent velocity field is considered. The problem is essentially equivalent to that in the theory of heat conduction of determining the temperature variation in a body when the initial temperature distribution is given. The long Section VI deals first with the problem of wave propagation. The results derived here are somewhat less general than those obtained and discussed by van de Hulst [Problems of Cosmical Aerodynamics, International Union of Theoretical and Applied Mechanics and International Astronomical Union, 1951; these Rev. **13**, 399]. Next the propagation of surface waves in a conducting fluid under the influence of gravity and in the presence of a magnetic field acting in the direction of the vertical is considered. It is shown that the velocity of propagation of these waves is given by $(hg+H^2/4\pi\rho)^{\frac{1}{2}}$ where g denotes the value of gravity and h the depth of the vessel (assumed small compared with the wavelength of the disturbance); arguments are advanced to support the view that these surface waves will be highly damped. Finally, the equations governing isotropic turbulence in magneto-hydrodynamics are also derived and briefly discussed.

S. Chandrasekhar (Williams Bay, Wis.).

***Oswatitsch, Klaus. Gasdynamik.** Springer-Verlag, Wien, 1952. xiv+456 pp. (3 plates). DM 78.00, $18.60.

The intent in this well-filled volume has been to present matter primarily of physical and technical interest, rather than mathematical. The subject, "gasdynamics", is one that extends well into the fields of physics, fluid mechanics, aerodynamics, and thermodynamics. After a thermodynamical introduction, steady and unsteady one-dimensional flows are treated in some detail, the latter by the characteristics technique. There follow two chapters devoted to general theorems and laws, and important applications of these. The inviscid-fluid approximation is then introduced and the greater part of subsonic, transonic, supersonic, and hypersonic aerodynamics, including the hodograph solutions, is briefly reported. Two closing chapters are devoted to flow with friction and experimental techniques (including analogies), respectively.

Throughout, limited space is given to detailed mathematical developments. More often the treatment consists of a description, in words, of the physical phenomena, together with the pertinent basic formulas (e.g., the differential equations) and typical results, presented analytically, graphically, and/or in tables. References to published literature, and even to the unpublished "sub-literature," are given in abundance (although the reviewer notes that there are some important American publications with which the author is apparently not familiar).

Actually, the scope of the work is impressive. Within the broad headings mentioned above, for example, the following rather specialized matters are discussed, as well as more familiar subjects: dissociation and ionization in shock waves, condensation shock, the interior-ballistics problem, behavior of cylindrical and spherical shocks near their centers, relation of drag to entropy, "Carnot shock", detonation, ram-jets, turbojets, rockets, relaxation methods, anisotropic plane and axisymmetric flow, boundary-layer flow and shock-wave interaction, and transonic wind tunnels. There are several useful tables of formulas and integrals, and a diagram for use in the characteristics technique. By design, questions of heat transfer are omitted, as are problems with appreciable effects of gravity (meteorological problems) and typically acoustical problems. Moreover, the medium is always a continuum.

The reader must recognize that this is no textbook; it is an encyclopaedic treatise of 470 rather crowded pages, in the German tradition, and as such it collects between its covers an immense amount of useful material.

W. R. Sears (Ithaca, N. Y.).

Hida, Kinzō. On the subsonic flow of a compressible fluid past a prolate spheroid. J. Phys. Soc. Japan **8**, 257–264; errata, 434 (1953).

The Janzen-Rayleigh method of expansion in powers of M^2, where M is the stream Mach number, is applied to the problem described by the title. The velocity potential is

calculated to the first approximation; i.e., terms in M^2 are obtained. Two numerical cases are worked out in detail, namely spheroids of thickness ratio 0.9 and 0.1. For the former, the velocity distribution on the surface is determined and plotted in comparison with related results. For the latter, only the maximum velocity, at the midsection, is found. In both cases the critical Mach number is determined for flow of air. In the first case, the result is 0.615, as compared with 0.587 for a sphere. In the second case, the value 0.972 is obtained, whereas linearized theory [e.g., Sears, Quart. Appl. Math. 5, 89–91 (1947); these Rev. 8, 540] gives the value 0.962.

A table of Legendre functions $P_n(z)$ and $Q_n(z)$ is appended, for $n=1, 2, \cdots, 13$ and selected values of z between 1 and 2.

W. R. Sears (Ithaca, N. Y.).

Martin, M. H. The propagation of a plane shock into a quiet atmosphere. Canadian J. Math. 5, 37–39 (1953).

In one-dimensional unsteady flow of a gas the velocity u, density ρ, and pressure p must satisfy the conditions:

$$\rho(u_x u+u_t)+p_x=0, \quad (\rho u)_x+\rho_t=0.$$

The author defines two functions ξ, ψ of x, t by the relations

$$(*) \qquad d\xi=\rho u dx-(\rho u^2+p)dt, \quad d\psi=\rho dx-\rho u dt.$$

ψ is a constant along a trajectory $x=x(t)$ of a gas particle in the (x, t)-plane and is therefore termed the trajectory function. The system (*) can be replaced by

$$(**) \qquad d\xi=ud\psi+tdp, \quad d\psi=\rho dx-\rho u dt, \quad \xi=\bar{\xi}+pt.$$

The introduction of ψ, p as independent variables into the system (**) yields the Monge-Ampère partial differential equation

$$\xi_{\psi\psi}\xi_{pp}-\xi^2_{\psi p}=\tau_p.$$

The arbitrariness in the function $\tau(\psi, p)$ may be limited by further hypotheses of a physical nature, e.g., $\tau=\delta(\psi)p^{-n}$, where $\delta(\psi)$ is an arbitrary function. The author proves the following theorem for a polytropic gas: Once the entropy distribution function $S(\psi)$ is specified, the determination of a condensation shock moving into a quiet atmosphere (and the flow immediately in back of it) reduces to the solution of a problem of Cauchy for the Monge-Ampère partial differential equation

$$\xi_{\psi\psi}\xi_{pp}-\xi^2_{\psi p}+n\delta(\psi)p^{-n-1}=0.$$

M. Pinl (Dacca).

Miles, John W. A general solution for the rectangular airfoil in supersonic flow. Quart. Appl. Math. 11, 1–8 (1953).

This is a solution of the problem of a rectangular wing tip; i.e., the quarter-infinite rectangular wing, in unsteady motion, arbitrarily distributed, under the approximations of the linearized theory. The acoustic equation is attacked by means of a Lorentz transformation and then Laplace transforms are introduced, assuming at this point that the motion is harmonic in terms of the Lorentz-transformed time variable. Thus, expressions for the velocity potential for such motion are obtained. Generalization to arbitrary time dependence is accomplished by Fourier transformation, and the result is put back into the original, physical coordinates. It consists, in general, of the quasi-steady potential plus a correction term in the form of a quadruple definite integral involving the prescribed downwash distribution $w(x, y, t)$. The special case of harmonic oscillation is written out. It is noted that an extension of Evvard's "equivalent area" concept is possible for the rectangular tip but not for oblique or curved edges. The extension of the present theory to the case of an oblique (subsonic leading) edge is indicated. In closing, the author mentions the analogous problem of diffraction and compares the various methods used there with wing-theory techniques. The relation of the present method to an extension of Lamb's method is also pointed out.

W. R. Sears (Ithaca, N. Y.).

Krasil'ščikova, E. A. Supersonic flow about thin bodies. Moskov. Gos. Univ. Učenye Zapiski 154, Mehanika 4, 181–239 (1951). (Russian)

This synthesizes or gives detailed proofs of results previously reported elsewhere [these Rev. 9, 392; 10, 77; 12, 216, 767; 13, 507]. Part I determines the velocity potential function for linearized supersonic flow over a harmonically oscillating and deforming thin wing, considering the influence of the wing's edges and trailing vortex sheet where necessary. As described in previous reviews, the potential is expressed as a power series in $\lambda^2=\omega^2a^2/(u^2-a^2)^2$ with double integral coefficients, where ω is the frequency, a the speed of sound, and u the speed of the undisturbed steady flow. The author outlines in detail the steps required to deal with wings of low aspect ratio (or for low supersonic speed) and with wings of non-convex plan-forms having areas cut out of their leading edges. She also describes a method for taking thickness into account. Part II deals with flow over non-oscillating wings, for which the potentials reduce to double integrals. Depending on where the potential is evaluated, the author simplifies the domain of integration and otherwise transforms the integrals, especially for wings of low aspect ratio. The pressure distribution on the wing is expressed in terms of surface integrals and line integrals along segments of the leading edge or characteristics. In some cases there may be curves of zero pressure on the wing. For plane wings with supersonic leading edges and straight tips, which need not be parallel, these curves are similar to the leading edge.

J. H. Giese (Havre de Grace, Md.).

Fjørtoft, Ragnar. Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. Geofys. Publ. Norske Vid.-Akad. Oslo 17, no. 6, 52 pp. (1950).

A detailed application is made of integral theorems, chiefly connected with energy, to the derivation of stability criteria for a stationary axially-symmetric vortex in the atmosphere. If K_m, ϕ and K_z are, respectively, the total kinetic energy of meridional flow, the total potential energy and the total kinetic energy of the zonal flow, then the stability, instability and kinematically conditioned instability are found from $K_m+\phi+K_z=$ constant, and conditions for minimum, maximum, and "saddle-point" extremum of $\phi+K_z$. A homogeneous fluid enclosed within fixed boundaries symmetrical with respect to the z-axis is next considered, axial symmetry being abandoned. The author next turns to the general non-axial-symmetric perturbations of an incompressible balanced vortex. Unstable waves are found for a baroclinic vortex.

G. C. McVittie (Urbana, Ill.).

Thompson, Philip Duncan. Notes on the theory of large-scale disturbances in atmospheric flow with applications to numerical weather prediction. Air Force Cambridge Research Center, Cambridge, Mass. Geophysical Research Papers, no. 16, 106 pp. (1952).

A procedure of integration of the physical equations of meteorology is discussed which is based on an atmospheric

model essentially the same as that used by Charney and Eliassen [Tellus 1, no. 2, 38–54 (1949); these Rev. 12, 555]: With barotropic flow and geostrophic vorticity, the vorticity equation reduces to a third-order differential equation in the stream function of the vertically integrated flow as function of the horizontal coordinate and time. Whereas Charney and Eliassen solve this nonlinear equation by numerical methods, the equation is linearized in this paper by assumption of small perturbations, and the solution obtained for a finite period in terms of a Green's function. The application to forecasting large-scale flow at the 500 mb level yields satisfactory results. *H. Panofsky.*

Haurwitz, B., and Craig, Richard A. Atmospheric flow patterns and their representation by spherical-surface harmonics. Air Force Cambridge Research Center, Cambridge, Mass. Geophysical Research Papers, no. 14, 78 pp. (1952).

Spherical surface harmonics have been applied to represent pressure and temperature patterns at constant levels. Observations were sufficiently complete for monthly normals only to be represented over the whole earth; daily and five-day mean patterns could be fitted in restricted bands in the northern hemisphere only. The results indicate that spherical harmonics give good representations of the observed pressure and temperature distribution; that the mean monthly pressure field computed from the temperature field resembles the actual distribution; but a simple forecasting technique based on the application of the vorticity equation to the pressure field under assumption of small perturbations of steady flow is not satisfactory. *H. Panofsky.*

Charney, J. G., and Phillips, N. A. Numerical integration of the quasi-geostrophic equations for barotropic and simple baroclinic flows. J. Meteorol. 10, 71–99 (1953).

Miles, John W. On acoustic diffraction through an aperture in a plane screen. Acustica 2, 287–291 (1952).

"The present paper is a condensation of a more extensive investigation undertaken in 1947, the publication of which (in its entirety) would have been superfluous after the appearance of the definitive papers of Levine and Schwinger" [Physical Rev. (2) 74, 958–974 (1948); 75, 1423–1432 (1949); these Rev. 10, 221, 764] and Levine [J. Acoust. Soc. Amer. 22, 48–55 (1950); these Rev. 11, 482]. In the author's opinion, it is of advantage to present the variational formulation via Fourier transforms as is done in the paper under review. Practically no new results are given.

C. J. Bouwkamp (Eindhoven).

Miles, J. W. On the diffraction of an acoustic pulse by a wedge. Proc. Roy. Soc. London. Ser. A. 212, 543–547 (1952).

Application of the method of "conical flow" to the problem referred to in the title. As the author states in a "note added in proof", he was anticipated by J. B. Keller and A. Blank [New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM-21 (1950); see also Comm. Pure Appl. Math. 4, 75–94 (1951); these Rev. 12, 564; 13, 304]. *C. J. Bouwkamp.*

Copson, E. T. On sound waves of finite amplitude. Proc. Roy. Soc. London. Ser. A. 216, 539–547 (1953).

The one-dimensional motion of a gas obeying the adiabatic law $p=k\rho^\gamma$ leads to the differential equation

$$(1)\qquad \frac{\partial^2 w}{\partial r\partial s}+\frac{N}{r+s}\left(\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}\right)=0,$$

where r, s are the Riemann invariants and

$$N=(3-\gamma)/(2\gamma-2).$$

The author observes that (1) is satisfied by integrals of the form

$$(2)\qquad w=\frac{1}{2\pi i}\int_C \frac{z^{2N}\phi(z)dz}{(z-r)^N(z+s)^N}$$

where C is a closed curve surrounding a cut containing the points 0, r, $-s$, outside of which the integrand is single-valued. If $\phi(z)$ is analytic along the real axis and even in z, the expression (2) corresponds to a motion starting from rest. Here ϕ can be chosen explicitly in such a way that initially x reduces to a prescribed function $x_0(r)$, provided this function is analytic for all real r and even. The results are applied to the problem of the expansion of a gas cloud into a vacuum. *F. John* (New York, N. Y.).

Magnus, Wilhelm. Infinite matrices associated with diffraction by an aperture. Quart. Appl. Math. 11, 77–86 (1953).

The theory of the diffraction of harmonic plane waves of sound incident normally on a plane screen with a circular aperture depends on finding a function $u(\rho, z)$ in cylindrical coordinates which is the solution of $\nabla^2u+k^2u=0$ satisfying Somerfeld's radiation condition at infinity and the conditions $u=0$ for $z=0$, $\rho>a$, $\partial u/\partial z=1$ for $z=0$, $0\leqq\rho<a$. If one can find the value $\Phi(\rho)$ taken by u for $z=0$, $0\leqq\rho<a$, then u can be found everywhere.

Levine and Schwinger [Physical Rev. (2) 74, 958–974 (1948); these Rev. 10, 221] showed that the ratio of the energy transmitted through the aperture to the energy incident on it is the imaginary part of the complex transmission coefficient T^* which is the quotient of two integrals involving $\Phi(\rho)$ quadratically. Their powerful variational method depends on the fact that T^*, regarded as a functional of Φ, is stationary for the correct function Φ. Moreover, they found approximate values for T^* by expanding $\Phi(\rho)$ as an infinite series of auxiliary functions with coefficients D_m; then T^* is stationary when these coefficients satisfy a certain inhomogeneous system of infinitely many linear equations with coefficient matrix L. Levine and Schwinger solved this system approximately "sectionwise".

The author now shows that it is possible to factorise L as a product $L^{(0)}S$, where $L^{(0)}$ is the matrix L for the special case $k=0$, and where S can be inverted by solving finite recurrence formulae. This algebraic property of L makes it possible not only to find approximate values of T^* but also to determine $\Phi(\rho)$. *E. T. Copson* (St. Andrews).

Elasticity, Plasticity

Pailloux, Henri. Quelques applications du calcul fonctionnel à la mècanique rationnelle. Ann. Sci. Ecole Norm. Sup. (3) 69, 213–257 (1952).

If f is a functional of a function x^P of variables P, and if the variation of x^P in the domain D gives a variation $\delta f=\int_D g_P\delta x^P d\tau$, where $d\tau$ is an element of D, then g_P is the functional derivative of f with respect to x^P, and is denoted in this paper by $\partial f/\partial x^P$. In this way the equations of equilibrium of an elastic body, for example, can be written

$$\frac{\partial \varpi}{\partial u}=X,\quad \frac{\partial \varpi}{\partial v}=Y,\quad \frac{\partial \varpi}{\partial w}=Z,$$

where ϖ is the energy, (u, v, w) the displacement, and (X, Y, Z) the body force. Apart from the use of this notation, the paper is best described by saying that it consists in the application of variational methods to problems in the mechanics of continua. The problems treated include the motion of an inextensible thread on a plane and on a surface, the vibrations of an elastic rod of any form and section, and the equilibrium of a beam, nearly straight and of slowly varying section, under the action of body forces. In this last problem the method is approximate, the displacement being assumed to be linear in the two coordinates in the plane perpendicular to the axis of the beam. The nine coefficients in the expressions for the displacement are functions of the coordinate in the direction of the axis, and the variational condition gives nine linear differential equations for them. The result is a formula for the bending of a beam which differs from the classical formula $EI/\rho=-M$ by the addition of a term on the right proportional to the transverse load and by the change of E to $E(1-\sigma)(1+\sigma)^{-1}(1-2\sigma)^{-1}$, where σ is Poisson's ratio. [The functional derivative, as used in this paper, appears to be equivalent to the variational derivative [Th. de Donder, Théorie invariantive du calcul des variations, nouvelle éd., Gauthier-Villars, Paris, 1935, p. 8]. The variational derivative, usually written with δ instead of ∂, has the advantage of being explicitly defined, once for all. The author indeed at one point obtains this explicit form for a particular case, but makes no reference to the variational derivative by name. No references to literature are included in the paper.] *J. L. Synge.*

***Sokolnikoff, I. S. On the use of conformal mapping in two-dimensional problems of the theory of elasticity.** Construction and applications of conformal maps. Proceedings of a symposium, pp. 71–77. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

This synoptic paper, closely related to a previous paper of the author [Bull. Amer. Math. Soc. **48**, 539–555 (1942); these Rev. **4**, 122], is intended to be a brief survey of some applications of conformal mapping to a class of two-dimensional boundary value problems in the theory of elasticity. Two problems are considered: To determine the state of stress and deformation of an elastic medium when the boundary is subjected to the action of prescribed forces and when the boundary points are subjected to prescribed displacements.

In case of an isotropic medium, represented on the complex $z=x+iy$-plane, Airy's stress function $U(x, y)$ satisfies the biharmonic equation $\Delta^2U=0$, of which the general solution is represented in the form $U=\Re(\bar{z}\varphi(z)+\chi(z))$, $\varphi(z)$ and $\chi(z)$ being analytic functions. In view of the boundary conditions, the determination of these functions is reduced to solving the functional equations, valid along the boundary, of the form $\varphi(z)+z\overline{\varphi'(z)}+\overline{\chi'(z)}=F(z)$ and $k\varphi(z)-z\overline{\varphi'(z)}-\overline{\chi'(z)}=G(z)$, respectively, $F(z)$ and $G(z)$ being the known functions and k a constant. If the basic region is the unit circle, then the formal solution of either problem can be obtained by means of the method of undetermined coefficients for power series or Fourier expansions. If, in general, the basic region is a simply connected domain bounded by a rectifiable Jordan curve, the problems are reducible to this simplest case by making use of the analytic function mapping the region onto the unit circle. The author further gives another method of attack on the problems which reduces them to solving an integral equation of Fredholm type.

In case of anisotropic medium, the differential equation satisfied by the stress function $U(x, y)$ has, in general, the general solution of the form $U=\Re(F_1(z_1)+F_2(z_2))$, $F_1(z_1)$ and $F_2(z_2)$ being analytic functions of the respective variables $z_1=x+\mu_1y$ and $z_2=x+\mu_2y$, where μ_1 and μ_2 are distinct complex (not real) constants. The author then states that if the given region R is simply connected, then the corresponding regions R_1 and R_2 in the z_1- and z_2-planes can be mapped conformally onto the unit circle in the ζ-plane with the aid of the mapping functions $z_1=\omega_1(\zeta)$ and $z_2=\omega_2(\zeta)$ in such a way that the correspondence of the boundary points of the regions R, R_1, and R_2 is preserved. Contrary to the author's assertion, it may, however, be readily shown that the preservation of the boundary correspondence in the stated manner is possible only if $\mu_1=\mu_2$. *Y. Komatu.*

Kennard, E. H. The new approach to shell theory: circular cylinders. J. Appl. Mech. **20**, 33–40 (1953).

The "new approach," based on "Epstein's basic work on . . . plates and shells," does not appear to differ essentially from Basset's method [Philos. Trans. Roy. Soc. London. Ser. A. **181**, 433–480 (1890); see also Hildebrand, Reissner, and Thomas, NACA Tech. Note no. **1833** (1949); these Rev. **11**, 69]. The speculations regarding "consistency" in approximation seem to the reviewer even less compelling than in most other articles on the theory of shells. The formulae for the various static resultants appear to differ somewhat from the numerous other proposals in the literature. In discussing special cases the author uses an argument equivalent to the following: In a theory where terms of second and higher order in a small parameter ϵ are to be neglected, suppose we are confronted with the equation $u_x=\epsilon u_{xx}$; then $u_{xx}=\epsilon u_{xxx}$, and hence $u_x=\epsilon^2u_{xxx}$, but now the right-hand side is of the order neglected, so $u_x=0$. This ingenious device for cutting the Gordian knot of a differential equation appears to be novel. *C. Truesdell.*

Csonka, P. Beitrag zur Theorie der elastischen Kreiszylinderschale. Acta Tech. Acad. Sci. Hungar. **6**, 167–176 (1953). (Russian summary)

The problem of bending of an elastic cylindrical shell of constant thickness was solved approximately by A. A. Jakobsen by means of an iterative method [Bauingenieur **20**, 394–405 (1939)]. In this paper an exact solution is found by a stress function $F(x, \varphi)$, whose derivatives express the components of displacements and stresses. The stress function has to satisfy a linear differential equation of the eighth order in x and φ. From the exact solution approximate formulas can be derived, which are in closer agreement with the exact solution than the iterative solution of Jakobsen. In the author's opinion the method represented renders the method of iteration superfluous. *R. Gran Olsson.*

Schäfer, Manfred. Über eine Verfeinerung der klassischen Theorie dünner schwach gebogener Platten. Z. Angew. Math. Mech. **32**, 161–171 (1952). (German. English, French and Russian summaries)

The author presents an alternate derivation of some recent results on the subject of small transverse bending of thin elastic plates. The theory in question permits the satisfaction of three conditions along the edge of a plate instead of the two conditions of the theory characterized by the differential equation $D\nabla^2\nabla^2w=p$. The method of the author combines assumptions concerning the distribution of stresses

across the thickness with suitable weighted averages taken over the stress-strain relations. In particular, weighted displacement averages are defined by equating the integrated work of the internal stresses to the work of the corresponding stress resultants and couples.

As an application of the theory the author considers a simply supported rectangular plate with the three boundary conditions of vanishing displacement, vanishing bending moment and vanishing twisting moment. By means of appropriate series solutions of the differential equations this problem is reduced to an infinite system of equations for an infinite number of series coefficients. This system is amenable to solution by iteration. The paper concludes with an approximate solution obtained in this manner and with a discussion of the qualitative behavior of this solution.

E. Reissner (Cambridge, Mass.).

Dean, W. R. The Green's function of an elastic plate. Proc. Cambridge Philos. Soc. **49**, 319–326 (1953).

In this paper a simple expression is found for the small transverse displacement of a thin isotropic elastic plate due to a transverse force applied at an arbitrary point of the plate. Rectangular axes $0x$, $0y$ are taken in the plane of the plate, and it is clamped along the semi-infinite lines $x\geqq 1$, $y=0$ and $x\leqq -1$, $y=0$. The transverse force is applied at some point (x_0, y_0) of the plate and the transverse displacement w at any point (x, y) is a biharmonic function of (x, y) which vanishes together with its normal derivative at all points of the boundary. It is thus the Green's function associated with the differential equation and the boundary conditions. Numerical values for w are calculated at points of $0y$ for three cases in each of which the force is applied at a point of $0y$.

R. M. Morris (Cardiff).

Chandra Das, Sisir. Note on the elastic distortion of a cylindrical hole by tangential tractions on the inner boundary. Quart. Appl. Math. **11**, 124–127 (1953).

This paper is concerned with the displacement produced in an isotropic elastic solid in the following three problems. (1) An infinite elastic plate having finite thickness with a cylindrical hole acted upon by a tangential traction localised within a zone of the hole less than the thickness of the plate, the surfaces of the plate being free from traction. (2) A similar plate to (1), but with one face of the plate fixed and a uniform tangential traction acting throughout the hole. (3) An infinite solid having a cylindrical hole acted upon by a tangential traction over a narrow band. The solutions in the first two cases are obtained in terms of infinite series of Bessel functions, while in the last case it is expressed as an infinite integral.

R. M. Morris (Cardiff).

Woinowsky-Krieger, S. Über die Anwendung der Mellin-Transformation zur Lösung einer Aufgabe der Plattenbiegung. Ing.-Arch. **20**, 391–397 (1952).

A plate having the form of an infinite sector is clamped along the edge $\theta=0$, free along $\theta=\alpha$ except for a point load of magnitude P at (r_0, α). Deflection of the plate is found with the aid of the Mellin transformation [as in I. N. Sneddon, Fourier transforms, McGraw-Hill, New York, 1951, p. 440; these Rev. **13**, 29]. The solution, expressed in integral form for the general case, is evaluated by numerical integration for $\alpha=\pi/4$ and $\alpha=\pi/2$, and by residues for $\alpha=\pi$. Further discussion concerns generalization of the method of loading.

R. E. Gaskell (Seattle, Wash.).

Woinowsky-Krieger, S. The bending of a wedge-shaped plate. J. Appl. Mech. **20**, 77–81 (1953).

An infinite, wedge-shaped, thin, elastic plate, bounded by the radial lines $\theta=-\alpha$ and $\theta=\alpha$, is bent by a single point load located at an arbitrary interior point $r=r_0$, $\theta=\theta_0$, $|\theta_0|<\alpha$. A general solution is obtained for the deflection with arbitrary homogeneous boundary conditions on the edges $|\theta|=\alpha$ (in particular, for either edge clamped, free or simply supported). The solution is carried out for $\theta_0=0$ and both edges clamped. The stress distribution along the edges and near the corner is investigated for $\alpha=\pi/4$, $\pi/2$ and π. The problem of a cantilevered infinite wedge loaded along the unclamped boundary was investigated by the author in the paper reviewed above.

W. Nachbar.

Al'perin, I. G. The stresses in an infinite strip uniformly compressed over half its length. Učenye Zapiski Har'kov. Gos. Univ. **28**, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) **20**, 107–118 (1950). (Russian)

The author solves the boundary-value problem in the title by a method originally used by O. M. Danilevs'kiĭ [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) **13**, 83–91 (1936)]. The boundary value problem consists in the determination of the real-valued Airy stress function $\varphi(x, y)$, which satisfies the biharmonic equation

$$\Delta\Delta\varphi=0 \quad\text{on}\quad -\infty<x<+\infty,\quad |y|<b,$$

plus the following stress boundary conditions:

$$\sigma_y=\frac{\partial^2\varphi}{\partial x^2}=0,\qquad\text{on}\quad y=\pm b,\quad 0<x<+\infty,$$

$$\tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=0,\quad\text{on}\quad y=\pm b,\quad -\infty<x<+\infty,$$

the following displacement boundary conditions:

$$v=\mp V_0,\quad -\infty<x\leqq 0,\quad y=\pm b$$

and the following conditions at infinity:

$$\varphi,\quad \frac{\partial^2\varphi}{\partial x^2},\quad \frac{\partial^2\varphi}{\partial y^2},\quad \frac{\partial^2\varphi}{\partial x\partial y}\to 0,\quad x\to+\infty,\quad |y|\leqq b,$$

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}\to 0,\quad \tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}\to 0,\quad x\to-\infty,\quad |y|\leqq b.$$

The method consists in writing

$$\varphi(x, y)=\int_{-i\infty}^{(0-),+i\infty}\frac{2K(mb)}{m^2}\cdot\frac{(\sin mb+mb\cos mb)\cos my+my\sin mb\sin my}{2mb+\sin 2mb}e^{mx}dm,$$

where, for each fixed complex number mb, the integrand is a solution of the biharmonic equation on the strip $-\infty<x<+\infty$, $|y|<b$, satisfying the boundary conditions (on $y=\pm b$, $-\infty<x<+\infty$)

$$\tau_{xy}=0,\quad \sigma_y=Ke^{mx},\quad Ev=\pm\frac{4K}{m}\frac{\sin^2 mb}{2mb+\sin^2 mb},$$

(E is Young's modulus), and then constructing the analytic function $K(mb)$ in order to satisfy the boundary conditions of the original boundary-value problem. The exact nature of the singularity of the solution at the points $x=0$, $y=\pm b$ is ascertained.

J. B. Dias (College Park, Md.).

Horvay, G. The end problem of rectangular strips. J. Appl. Mech. 20, 87–94 (1953).

The problem of a long and narrow rectangular strip, loaded on the four sides by a self-equilibrating system of edge tractions, can often be split into the sum of several subsidiary problems, of which all but one have been considered at length in the literature. This one is the problem of the strip with the long sides free of stress and the narrow ends each loaded by surface tractions having zero resultant force and moment. The stresses contributed by this problem can be ignored in the sum for many applications by invocation of St. Venant's Principle. This paper considers the effect of these stresses more precisely, however, by treating the problem of the semi-infinite rectangular strip, free along the sides, and loaded on the narrow edge by an arbitrary system of self-equilibrating tractions. The analysis of the semi-infinite strip will apply to the long strip provided that the interaction of end conditions from the two ends can be neglected; the author derives as one result estimates of the depth of penetration of the surface tractions and of minimum length of strip to insure that interaction effects can be neglected.

Successive approximations to the solution, as determined by the Airy function $\phi(x, y)$, are obtained by a minimum energy method, assuming a representation in the form

$$\phi(x, y) = \sum_k c_k\phi_k = \sum_k c_k f_k(y) g_k(x), \quad -1 \leq y \leq 1, \quad x \geq 0;$$

$f_k(y)$ are a complete set of orthogonal polynomials which are appropriate for the homogeneous boundary conditions at $y=\pm1$, and each of which represents a system of self-equilibrating stresses at $x=0$. The series $\sum c_k f_k$ is obtained by expansion of the given boundary value $\phi(0, y)$ of the Airy function. A table of the first nine polynomials f_k, as well as the recursion formula and differential equation that they satisfy, are given. The functions $g_k(x)$ are determined for the first approximation from the Euler-Lagrange equations derived by requiring that each ϕ_k separately minimize the strain energy; higher approximations are then also determined. The ϕ_k so determined are not biharmonic functions, but the rapid convergence of higher approximate eigenvalues to the actual biharmonic eigenvalues is indicated.

The otherwise lucid presentation in this paper was marred for the reviewer by the employment of an unnecessarily complicated system of symbols and by the use of an unorthodox and misleading notation for inner product.

W. Nachbar (Seattle, Wash.).

Livesley, R. K. Some notes on the mathematical theory of a loaded elastic plate resting on an elastic foundation. Quart. J. Mech. Appl. Math. 6, 32–44 (1953).

The problem of a concrete slab, supported on an earth foundation, and subjected to load applied over its upper face is of considerable technical importance in the design of paving slabs for roads and aircraft runways. This problem has been extensively discussed with the object of providing the engineer with practical design formulae. The translation of the physical problem into a mathematical problem whose solution is tractable requires much idealization, and the usefulness of theoretical results must be checked, whenever possible, through their direct comparison with experimental results. Fortunately, in many particular situations, experience has shown that it is quite satisfactory to treat the slab as a thin, elastic, homogeneous, isotropic plate which, under applied load and foundation reaction the latter at any point being assumed proportional to the local transverse deflexion, behaves as postulated in elementary plate theory.

The author reviews the physical problem, and then discusses the solution of the mathematical problem, involving the above simplifications, in certain particular instances. For simplicity, the procedure is often formal, and accordingly exact conditions of validity of the solutions constructed are not then given. There still remain interesting and important problems in this field whose solution would well merit future attention.

Here, there is first discussed in detail the case in which the loading is static, and attention is given both to infinite and to semi-infinite rectangular plates. Fourier integrals are used to solve problems in which all plate edges are simply supported. The difficulties that arise when other edge conditions exist are discussed, and a method is indicated for dealing with clamped edges. The problem of a semi-infinite rectangular plate under a given distribution of shear and bending moment along its free boundary is solved. Second, there is discussed in detail the problem which involves a constant applied load travelling with uniform velocity across the surface of an infinite plate. It is shown that there exists a certain critical velocity beyond which stresses and deflexions become infinite. The ratios of the maximum deflexion and bending moment to their static values are expressed as functions of λ, the ratio of the actual velocity to its critical value. It is found that although these deflexions and stresses are greater than their static values, the increase is small unless λ approaches unity. *H. G. Hopkins.*

Solyanik-Krassa, K. V. On the solution of an axially symmetric problem of the theory of elasticity. Doklady Akad. Nauk SSSR (N.S.) 86, 481–484 (1952). (Russian)

The calculation of the stresses in an axially symmetric body under an axially symmetric load is reduced to the calculation of a function φ which satisfies the equation (in cylindrical coordinates)

$$\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\right)\right\}\varphi=0$$

and a function Φ which satisfies the equation

$$\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\right)\right\}^2\Phi=0,$$

both with suitable boundary conditions. Because these equations are invariant for conformal transformations of the complex $(r+iz)$ plane, coordinates adapted to the boundary conditions can be chosen. In the new coordinates (ξ, η),

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\right) \quad \text{becomes} \quad \frac{\partial}{\partial \xi}\left(\frac{1}{r}\frac{\partial}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{1}{r}\frac{\partial}{\partial \eta}\right)$$

W. H. Muller (Amsterdam).

Ziegler, H. Kritische Drehzahlen unter Torsio und Druck. Ing.-Arch. 20, 377–390 (1952).

The author treats the critical velocities of rotating shafts for several types of mounting of shafts and location of attached wheel or disk. In addition to rotation the system receives axial torsion W and pressure D applied either at the end of the shaft or to the disk. The results are incorporated in the expression

$$\frac{\omega_k}{\omega_0}=1-k_1\frac{Dl^2}{EI}\pm k_2\frac{Wl}{EI}-k_3\left(\frac{Wl}{EI}\right)^2$$

where k_1, k_2, k_3 are constants tabulated for specific cases, l=length, EI=stiffness modulus, and ω_k/ω_0 is the ratio of the critical speed of the system to the critical speed of the unloaded system. A discussion is given showing what factors raise or lower the critical speeds. *D. L. Holl.*

***Dengler, M. A., and Goland, M. Transverse impact of long beams, including rotatory inertia and shear effects.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 179–186. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Based on the Timoshenko differential equation for the transverse bending of elastic beams, closed solutions are deduced for the stresses induced in a long beam of uniform section by the action of an impulsive, concentrated transverse load. *From the authors' summary.*

Rahmatulin, H. A. Normal impact on a flexible cord with variable velocity. Moskov. Gos. Univ. Učenye Zapiski **154**, Mehanika 4, 267–274 (1951). (Russian)

A mass point moves with variable velocity perpendicular to a stretched cord of infinite length. The shape of the cord is calculated by a linearised theory. This linearisation causes the velocity of the mass point to be a linear function of the time, and the shape of the cord becomes a quadratic function of the coordinate along the cord. *W. H. Muller.*

Neuber, H. Theorie der Druckstabilität der Sandwichplatte. I, II. Z. Angew. Math. Mech. **32**, 325–337 (1952); **33**, 10–26 (1953). (English, French and Russian summaries)

The author presents an analysis of the stability under unidirectional compressive load of a plate of sandwich construction composed of isotropic materials. Both types of possible instability, either face wrinkling or ordinary Euler buckling, are included. In the second part of the paper the formulas are simplified for practical applications under suitable assumptions and design curves are presented. For treatments of the problem which are not quite so complete from the theoretical standpoint, but which include sandwich constructions with orthotropic cores, reference may be made to papers by Williams, Leggett and Hopkins [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **1987** (1941)] and by H. L. Cox [ibid. no. **2125** (1946)]; also to a paper by the reviewer [Proc. Symposia Appl. Math. v. 3, pp. 85–106, McGraw-Hill, New York, 1950; these Rev. **12**, 771] containing the results of an approximate analysis, experimental results and further references to the literature. *H. W. March.*

Sretenskiĭ, L. N. The propagation of elastic waves arising from the motion of a system of normal stresses on the surface of a half-space. Trudy Moskov. Mat. Obšč. **1**, 167–186 (1952). (Russian)

L'auteur étudie la propagation des ondes suscitées dans un demi-espace élastique par un système de tensions superficielles se déplaçant de façon rectiligne avec une vitesse constante. Tel est le cas des microseismes provoquées par le déplacement sur la surface terrestre des régions cycloniques ou anticycloniques. En définissant les déplacements élastiques par les formules $u=u'-\partial\phi/\partial x$, $v=v'-\partial\phi/\partial y$, $w=w'-\partial\phi/\partial z$ avec ϕ, u', v', w' vérifiant les équations bien connues $\partial^2\phi/\partial t^2=(\lambda+2\mu)\rho^{-1}\Delta\phi$, $\partial u'/\partial x+\partial v'/\partial y+\partial w'/\partial z=0$, $\partial^2u'/\partial t^2=\mu\rho^{-1}\Delta u'$, $\partial^2v'/\partial t^2=\mu\rho^{-1}\Delta v'$, $\partial^2w'/\partial t^2=\mu\rho^{-1}\Delta w'$ et en donnant au tenseur de la tension la forme

$$X_x=-\lambda\Delta\phi-2\mu\frac{\partial^2\phi}{\partial x^2}+2\mu\frac{\partial u'}{\partial x},\cdots,$$

$$Y_z=Z_y=-2\mu\frac{\partial^2\phi}{\partial y\partial z}+\mu\left(\frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}\right),\cdots,$$

l'auteur cherche les solutions élémentaires sous la forme

$$\phi=D\exp[i(kx+my+\alpha t)+\gamma z],$$
$$u'=A\exp[i(kx+my+\alpha t)+sz],\cdots.$$

Les conditions superficielles

$$X_z=0,\quad Y_z=0,\quad Z_z=f\exp[i(kx+my+\alpha t)]$$

permettent de mener les calculs jusqu'au bout. L'auteur étudie le cas de tensions localisées dans une bande parallèle à l'axe des abscisses et le cas du système de pressions $P=T$ pour $(-ct-a<x<-ct+a)$ et nulle partout ailleurs. Dans tous ces cas on peut calculer les déplacements superficielles à une distance suffisamment grande de la trajectoire du système donné des tensions normales. *V. A. Kostitzin.*

Haskell, N. A. The dispersion of surface waves on multilayered media. Bull. Seismol. Soc. America **43**, 17–34 (1953).

Reviewed earlier in report form; cf. these Rev. **13**, 512.

Meier, Rudolf, und Schuster, Kurt. Zur Theorie der Schallausbreitung in piezoelektrischen Kristallen. Ann. Physik (6) **11**, 397–406 (1953).

Review of the Christoffel equations of elastic small amplitude plane wave propagation in anisotropic continua, with inclusion of the piezoelectric couplings via the "e" matrix (strain and field as independents); dielectric conditions: absence of free charge and irrotationality of the field vector ("elastic"≪"electromagnetic" phase velocities). Applications to special classes: D_3; C_2; V_d (Quartz; *LSH*; *ADP*) and special propagation directions: k_x; k_y; $k_x^2=k_y^2=k_z^2=\frac{1}{3}$ ("*L*-cut"), respectively. In the first two cases, the secular equation is reducible. Numerical discussion.

H. G. Baerwald (Cleveland, Ohio).

Thomas, T. Y. Singular surfaces and flow lines in the theory of plasticity. J. Rational Mech. Anal. **2**, 339–381 (1953).

The paper is concerned with stationary and moving surfaces of discontinuity ("singular surfaces") in a perfectly plastic solid that obeys the yield condition and flow rule of v. Mises. Presumably to make the paper self-contained, the author redevelops in considerable detail the kinematical conditions of compatibility for discontinuity surfaces of the orders zero, one, and two. The argument in this first part of the paper follows familiar lines [see, for instance, P. Appell, Traité de mécanique rationnelle, vol. 3, 3rd ed., Gauthier-Villars, Paris, 1921, chapter 33, III], but is rendered more transparent by the systematic use of tensor notation. To derive the dynamical conditions of compatibility for a perfectly plastic solid, the author uses the reduced system of equations derived in an earlier paper [J. Rational Mech. Anal. **1**, 343–357 (1952); these Rev. **14**, 113]; these equations are obtained by eliminating the components of the stress deviation from the complete system of equations of the Mises theory. A first type of discontinuity surface is stipulated to be of the order one with respect to pressure and density, but of the order two with respect to the velocity

components. It is found that one of the principal strain rates must vanish at any point of such a surface, the tangent planes of the surface being planes of maximum shear. The last remark identifies the considered type of discontinuity surface with a characteristic surface of the Mises equations. As the author has shown previously (see the paper cited above), such surfaces are not, in general, possible. In addition to these surfaces of weak discontinuity, slip surfaces are considered across which the tangential velocity components are discontinuous. To construct examples of discontinuity surfaces the author considers test specimens in which the state of stress is assumed to be nearly uniaxial. Assuming further that the normal particle velocity vanishes over the considered slip surface, and that the flow is approximately stationary on either side of this surface, the author shows that the slip surface must be a ruled surface whose rectilinear generators are stream lines along which the velocity vector is constant. Plane, conical, and helicoidal slip surfaces satisfying this condition are discussed in some detail. *W. Prager* (Providence, R. I.).

Kondo, Kazuo. A new theory concerning the yielding of materials based on Riemannian geometry. Jap. Sci. Rev. Ser. I. **1**, no. 3, 7–10 (1950).

A summary of work published earlier in Japanese [J. Soc. Appl. Mech. Japan **2**, 29–31, 146–151 (1949); these Rev. **11**, 289, 703].

Hill, R. On discontinuous plastic states, with special reference to localized necking in thin sheets. J. Mech. Phys. Solids **1**, 19–30 (1952).

Permissible discontinuities of stress, velocity, and surface slope are investigated in a plastic-rigid sheet deformed in its plane. One such discontinuity of velocity is shown to be the mathematical idealization of localized necking; the necessary restrictions on the stress-state and rate of work-hardening are obtained for any yield function and plastic potential. The results are illustrated by an examination of the modes of necking in notched tension strips. The constraint factors at the yield point are obtained for notches with wedge-shaped or circular roots. (From author's summary.) *H. G. Hopkins* (Providence, R. I.).

Seeger, Alfred, Donth, Hans, und Kochendörfer, Albert. Theorie der Versetzungen in eindimensionalen Atomreihen. III. Versetzungen, Eigenbewegungen und ihre Wechselwirkung. Z. Physik **134**, 173–193 (1953).

Im ersten und zweiten Teil dieser Arbeit [Kochendörfer und Seeger, Z. Physik **127**, 533–550 (1950); Seeger und Kochendörfer, ibid. **130**, 321–336 (1951); diese Rev. **12**, 304; **13**, 406] wird für die Bewegung von Versetzungen eine Differentialgleichung vom elliptischen Typ hergeleitet und deren Lösung besprochen. Der vorliegende dritte Teil befasst sich mit der Wechselwirkung von Versetzungen und Schallwellen. Es wird gezeigt, dass es trotz der Nichtlinearität des Kraftgesetzes, gewisse Kombinationen von thermischen Wellen gibt, welche an Eigenschwingungen erinnernde Eigenschaften haben. Mit Hilfe von Bäcklund-Transformationen, die eingehender besprochen werden, können einfache Gesetze für die Superposition von hier auftretenden Bewegungen angegeben werden. Die Wechselwirkungen derer mit Versetzungen werden besprochen. Es folgt, dass im eindimensionalen Modell keine Dämpfung von bewegten Versetzungen auftritt, es entsteht jedoch eine solche im dreidimensionalen Fall, bei der Berücksichtigung der atomistischen Struktur. *Th. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

***Wallot, Julius. Grössengleichungen, Einheiten und Dimensionen.** Johann Ambrosius Barth Verlag, Leipzig, 1953. viii+216 pp. DM 14.70; bound, DM 16.35.

A critical and philosophical discussion of various dialectics for units and dimensions. The underlying ideas were first stated by the author in the Elektrotech. Z. **43**, 1329–1333, 1381–1386 (1922). The arithmetic and algebraic relations between various systems of units in different branches of physics are described carefully, with many bibliographical references. The strictly mathematical content is not new. *G. Birkhoff* (Cambridge, Mass.).

Polvani, Giovanni. I fondamenti concettuali e teorici della metrologia fisica. Rend. Sem. Mat. Fis. Milano **22** (1951), 108–150 (1952).

The author summarizes his metaphysics of measurement, published in full in his book "Elementi di metrologia teoretica", Marzorati, Milan, 1947. His examples are mostly geometrical. No references are given to other authors. *G. Birkhoff* (Cambridge, Mass.).

Vermeulen, R. Dimensional analysis, units and rationalization. Philips Research Rep. **7**, 432–441 (1952).

An intelligent polemic attacking the indiscriminate manipulation of dimensional formulas, especially in electromagnetism. Various absurdities are traced to the loose use of the multiplication sign for "products" which have no specific physical meaning. *G. Birkhoff.*

Hoffmann, Banesh. The relativity of size. Physical Rev. (2) **89**, 49–52 (1953).

A change of scale is not included in the ordinary transformations of tensor calculus. The author asserts that physical theories should be invariant under such changes. In other words, the author claims, contrary to Eddington, that all physics is what Eddington, in his Fundamental Theory [Cambridge, 1946; these Rev. **11**, 144], calls "scale-free physics". *A. J. Coleman* (Toronto, Ont.).

Hoffmann, Banesh. The similarity theory of relativity and the Dirac-Schrödinger theory of electrons. Physical Rev. (2) **89**, 52–59 (1953).

Scale-free physics is investigated as a particular case of conformal physics. It is expressed in terms of a complicated variational principle, and a set of equations is obtained, of which the Dirac-Schrödinger equations for the "new classical theory of electrons" [Schrödinger, Nature **169**, 538 (1952); these Rev. **13**, 893] is a particular case. *A. J. Coleman.*

Optics, Electromagnetic Theory

Suchy, Kurt. Schrittweiser Übergang von der Wellenoptik zur Strahlenoptik in inhomogenen anisotropen absorbierenden Medien. I. Gleichungen für Wellennormale, Brechungsindex und Polarisation. Ann. Physik (6) **11**, 113–130 (1952).

After transforming the vector wave equation into the cartesian vector form with the aid of "curvature condi-

tions", the author derives the characteristic equation from Debye's transformation by using a vector phase function. Assuming two of the components of the phase function to depend linearly on the third, the characteristic equation reduces to a non-linear partial differential equation in the third component. Two special cases are considered: (a) when one of the components of the normal vector depends on the other components, and (b) when one introduces normal coordinates on the wave surface. For a stratified medium (case (a)), the characteristic equation becomes a non-linear ordinary equation of first order in one of the components of the normal vector, which possesses in general four solutions corresponding to two reflected and two incident waves. In case (b) the characteristic equation reduces to a non-linear partial differential equation in the refractive index function n,

$$\text{(a)}\quad aw^2+bw+c=0,\quad w=n^2+\frac{1}{ik_0h_1(v_j)}\frac{\partial n}{\partial v_1}\quad (j=1,2,3)$$

where a, b, c depend on the elements of the dielectric tensor. The two solutions of (1) account for the double refraction. For stratified media $h_1=h_1(v_1)$ and (a) reduces to an equation of Riccati's type. In the last section the author discusses the principal polarization ellipses of case (b).

N. Chako (Cambridge, Mass.).

Pohlack, Hubert. Zur Umkehrbarkeit der Lichtwege in geschichteten Medien. Ann. Physik (6) 11, 145–154 (1952).

The author considers the passage of a plane polarized wave normally incident on a set of plane parallel layers of plates each having different dielectric constants and conductivity. From the boundary conditions at the surface of each plate, the total transmission coefficient $D_{0,m+1}$ is obtained. If T is the transmission matrix, then $W_{in}=TW_t$ where W_{in} and W_t represent the incident and the transmitted wave respectively, and T is the product of the transmission matrices T_k of each layer. If the regions outside the plates are non-absorbing, then the transmission coefficient $D_{m+1,0}$ which is obtained by reversing the direction of propagation of the polarized wave is found to be equal to $D_{0,m+1}$ and the Helmholtz reciprocity rule is verified. However, if one or both outside regions (media) are absorbing, the transmission coefficients are not the same and Helmholtz rule breaks down. [Reviewer's remarks: The above problem has been treated in all its generality, i.e., for oblique incidence and for both cases when ϵ_k, σ_k, μ_k are constants and when they vary continuously within each layer by R. K. Luneberg, Mathematics Research Group, New York University, Report **172-3** (1947).]

N. Chako (Cambridge, Mass.).

Moon, Parry, and Spencer, Domina Eberle. Theory of the photic field. J. Franklin Inst. 255, 33–50 (1953).

In this paper the authors have listed a number of theorems of vector analysis which can be applied to photic field theory. Of particular interest are theorems 4, 7, and 9 on the existence of quasi-potential functions in terms of which the light-vector D is expressed. If the field is axially symmetric or is uniform along one coordinate axis, then it is shown that a quasi-potential function always exists. [Reviewer's remark: These results are well-known facts in the theory of normal congruences.] For a surface source of finite radius of curvature radiating uniformly in all directions, the authors have shown that the light-vector D is normal to it and attains the value of the helios H on the surface provided that the medium is homogeneous. By applying the above conditions to the light-vector, they have calculated the field produced by an oblate spheroidal surface source of uniform helios when the medium is non-dissipative. *N. Chako.*

Tallqvist, Hj. Einige Reflexions- und Refraktionsprobleme. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 7, 18 pp. (1951).

Given a surface S and two points A and B, the problem is to find that point (or those points) P on S such that a ray passes from A to B with reflection (or refraction) by S at P. The work consists of direct elementary applications of Fermat's principle, the media being homogeneous and isotropic. *J. L. Synge* (Dublin).

Vasseur, Jean Pierre. Diffraction des ondes électromagnétiques par des ouvertures dans les écrans plans conducteurs. Ann. Physique (12) 7, 506–563 (1952).

The purpose of this study is to examine the possibility of using the principle of Huyghens in the diffraction of plane electromagnetic waves by an opening in a plane conducting screen. Some formulae are obtained, which permit such an interpretation. Experimental results are also described.

A. E. Heins (Copenhagen).

Miles, J. W. On the diffraction of an electromagnetic pulse by a wedge. Proc. Roy. Soc. London. Ser. A. 212, 547–551 (1952).

The solution of this quasi-two-dimensional vector problem is reduced to a suitable linear superposition of two appropriate solutions of analogous scalar problems [cf. same Proc. **212**, 543–547 (1952); these Rev. **14**, 816].

C. J. Bouwkamp (Eindhoven).

Eckart, Gottfried, et Liénard, Pierre. Analogie incomplète des impedances caractéristiques électrique et acoustique et conséquences relatives à l'écho dans les milieux stratifiés continus. Acustica 2, 157–161 (1952).

Die Verff. gehen von den Maxwellschen Gleichungen für das Vakuum aus, definieren für eine ebene fortschreitende Welle eine charakteristische Impedanz des Vakuums, welche als Quotient der elektrischen und magnetischen Feldstärke erscheint. Im akustischen Falle betrachten Verfasser ebenfalls eine ebene Welle und gehen von den beiden Differentialgleichungen aus, welche den Druckgradienten mit der Geschwindigkeit und den Geschwindigkeitsgradienten mit dem Druck verknüpfen. Diese Differentialgleichungen sind formal analog zu den obengenannten Maxwellschen Gleichungen. Dementsprechend kann eine charakteristische akustische Impedanz definiert werden. Verff. gehen dann zu den Differentialgleichungen für eine ebene akustische Welle aus, welche in einem Medium verläuft, das durch die Schwerkraft veränderliche Eigenschaften hat. Sie zeigen, dass in einem solchen Medium stets innere Reflexion der akustischen Wellen stattfindet. Somit ist keine vollständige Analogie zwischen der charakteristischen Impedanz im elektrischen und im akustischen Falle vorhanden.

M. J. O. Strutt (Zürich).

Jones, D. S. The behaviour of the intensity due to a surface distribution of charge near an edge. Proc. London Math. Soc. (3) 2, 440–454 (1952).

The author studies the order of magnitude of certain field quantities in diffraction theory (electromagnetic or acoustic) in the neighborhood of the edge of a surface.

A. E. Heins (Copenhagen).

Saunders, William K. On solutions of Maxwell's equations in an exterior region. Proc. Nat. Acad. Sci. U. S. A. **38**, 342–348 (1952).

The author proves that the time-periodic Maxwell equations in a non-absorbing homogeneous and isotropic medium outside a sufficiently smooth finite closed surface C have one and only one solution satisfying the radiation condition at infinity and assuming given boundary values for the tangential electric field at C. The uniqueness theorem is an extension of Rellich's theorem for the scalar case [Jber. Deutsch. Math. Verein. **53**, 57–65 (1943); these Rev. **8**, 204] while the existence theorem is based on a recent paper by C. Müller [Math. Ann. **123**, 345–378 (1951); these Rev. **13**, 514]. *C. J. Bouwkamp* (Eindhoven).

Hafner, Erich. Das vollständige System der elektromagnetischen Eigenschwingungen einachsig anisotroper Hohlraumresonatoren. Arch. Elektr. Übertragung **7**, 47–56 (1953).

The steady-state solutions of Maxwell's equations in uniaxial anisotropic media of dielectric constant

$$\epsilon=\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_2\end{pmatrix}$$

are considered. Expressions for electric-type and magnetic-type fields that are cylindrical with respect to the direction of propagation are given in terms of the well-known scalar functions U and V [Bromwich, Philos. Mag. (6) **38**, 143–164 (1919); Borgnis, Ann. Physik (5) **35**, 359–384 (1939)]. These expressions are used to solve two cavity resonator problems possessing the proper cylindrical symmetry. One problem involves a cavity formed by a perfectly conducting circular cylinder with perfectly conducting end-plates and partially filled from one end-plate to an arbitrary cross-section with an uniaxial, anisotropic medium so oriented that ϵ_1 is the dielectric constant in the direction of the axis of the cylinder. The other problem consists of a cavity bounded by a perfectly conducting parallelepiped and partially filled in a similar manner. *C. H. Papas.*

Teichmann, T., and Wigner, E. P. Electromagnetic field expansions in loss-free cavities excited through holes. J. Appl. Phys. **24**, 262–267 (1953).

The authors place in evidence the elusive fact that it is impossible to expand the electromagnetic field of a dissipationless cavity with coupling apertures in terms of the proper vector functions of the cavity completely enclosed. They show that for completeness it is necessary to add an irrotational magnetic field to the set of magnetic vector functions [J. C. Slater, Microwave electronics, Van Nostrand, New York, 1950, Chapter 4]. The contribution of this field to the admittance matrix of the junction, i.e., cavity with apertures, is a term inversely proportional to the frequency. The acoustical counterpart of this problem can be found in a discussion of the Helmholtz resonator by Levine [J. Acoust. Soc. Amer. **23**, 307–311 (1951); these Rev. **13**, 305]. Another method of treating junction problems in great generality is based on a theory (unpublished, unfortunately) by Schwinger, which employs the dyadic or tensor Green's function [H. Levine and J. Schwinger, Comm. Pure Appl. Math. **3**, 355–391 (1950); these Rev. **13**, 305]. *C. H. Papas* (Pasadena, Calif.).

Iijima, Taizo. On the electromagnetic fields in case of existence of a semi-infinite hollow conductive circular cylinder. II. Electrotechnical Laboratory, Agency of Industrial Science and Technology, Tokyo, Rep. no. **531**, 214 pp. (1952). (Japanese)

The author studies the steady state axial symmetric electromagnetic field generated in a semi-infinite circular tube. The problem is reduced to the solution of integral equations of the Wiener-Hopf type and these in turn are solved by Fourier methods. From the Fourier transforms, the author gives the reflection coefficient and the radiation pattern in closed form. The results are too complicated to give in detail. *A. E. Heins* (Copenhagen).

Gamo, Hideya. The Faraday rotation of waves in a circular waveguide. J. Phys. Soc. Japan **8**, 176–182 (1953).

Microwave propagation is examined for a circular guide uniformly filled with a substance whose dielectric constant and permeability, i.e., whose macroscopic constitutive parameters, are given respectively by the matrices:

$$\begin{pmatrix}\epsilon & -i\alpha & 0\\ i\alpha & \epsilon & 0\\ 0 & 0 & \epsilon_z\end{pmatrix} \text{ and } \begin{pmatrix}\mu & -i\kappa & 0\\ i\kappa & \mu & 0\\ 0 & 0 & \mu_z\end{pmatrix}$$

where the rotational parameters α and κ are linear functions of the static magnetic field externally applied along the guide axis $0z$. Quantum mechanically α and κ are accounted for by the Zeeman splitting of the spectral lines and rotary dispersion associated with ferromagnetic and paramagnetic absorption respectively [D. Polder, Philos. Mag. (7) **40**, 99–115 (1949)].

The normally degenerate modes are split by the applied magnetic field into two circularly polarized component waves, one rotating clockwise and the other counterclockwise. And because these component waves travel with different velocities, a rotation of the field is produced. Only in the limit of vanishing applied magnetic field is it possible to have pure TE- and TM-modes and for this reason the modes are suggestively labeled quasi-TE or quasi-TM. [Cf. H. Suhl and L. R. Walker, Physical Rev. (2) **86**, 122–123 (1952).]

In this paper the author deduces general expressions for the field components and computes the cut-off frequencies of several quasi-TE and quasi-TM modes. He also computes the frequency dependence of the propagation constants for both component waves of the quasi-TE_{11} mode and the quasi-TM_{11} mode. *C. H. Papas* (Pasadena, Calif.).

Kacenelenbaum, B. Z. Wave guides with nonideal walls. Doklady Akad. Nauk SSSR (N.S.) **88**, 37–40 (1953). (Russian)

The author determines the velocity damping and field configuration of electromagnetic waves in a waveguide of arbitrary cross-section, the walls of which are good, but not perfect, conductors. His analysis is based on the successive expansion of the fields in powers of the complex wave resistance of the waveguide walls. This permits the application of Leontovich's boundary condition [Investigations into the Propagation of Radio Waves, Vol. 2, 1948 (in Russian)] and the consideration of only the interior region of the waveguide. *C. H. Papas* (Pasadena, Calif.).

Budden, K. G. The theory of the limiting polarization of radio waves reflected from the ionosphere. Proc. Roy. Soc. London. Ser. A. 215, 215–233 (1952).

It is well known that because of the earth's magnetic field, the ionosphere acts like a doubly refracting medium; consequently, when a radio wave is incident on the ionosphere, two different characteristic waves are propagated. This paper considers a single characteristic wave travelling downward after reflection in the ionosphere and discusses the limiting polarization attained by this limiting wave at the bottom of the ionosphere. If it is assumed that the waves are vertically incident on a horizontally stratified ionosphere with an oblique magnetic field, the relevant equations were given by Försterling [Hochfrequenztech. Elektroak. 59, 10–22 (1942)]. They reduce to the following set of coupled second-order differential equations:

$$(*)\qquad \begin{aligned} &\frac{d^2\pi_o}{dh^2}+k^2\mu_o{}^2\pi_o=f_1(\pi_o,\pi_e),\\ &\frac{d^2\pi_e}{dh^2}+k^2\mu_e{}^2\pi_e=f_2(\pi_o,\pi_e). \end{aligned}$$

Here the subscripts o and e refer to the "ordinary" and "extraordinary" characteristic waves, respectively. The functions π_o, π_e are the amplitudes of these waves while $\mu_o{}^2$, $\mu_e{}^2$ are the corresponding indices of refraction. Also, the right-hand sides are linear functions of π_o and π_e and their derivatives. The author solves these equations by a modification of the W.K.B. method. Assume that $u_o{}^{(1)}$, $u_e{}^{(1)}$ are the up-going and $u_o{}^{(2)}$, $u_e{}^{(2)}$ the down-coming solutions of (*) in case $f_1=f_2=0$. Then, for the case $f_1\neq 0$, $f_2\neq 0$, assume $\pi_o=A_ou_o{}^{(2)}$, $\pi_e=A_eu_e{}^{(2)}$, where A_o and A_e are slowly varying functions of h. By substitution in (*), the author finds $2ikdA_o/dh=u_o{}^{(1)}f_1$, $2ikdA_e/dh=u_e{}^{(1)}f_2$. The solution of these equations enables the author to obtain the limiting polarization of the characteristic waves. A reasonable model of the ionosphere is then assumed and it is shown that an experimental study of limiting polarizations at frequencies greater than 1 MC/sec is not likely to lead to any new information about the ionosphere. *B. Friedman.*

***Weber, Ernst. Conformal mapping applied to electromagnetic field problems.** Construction and applications of conformal maps. Proceedings of a sympsoium, pp. 59–69. National Bureau of Standards, Appl. Math. Ser., No. 18, U. S. Government Printing Office, Washington, D. C., 1952. $2.25.

The paper contains a synoptic survey indicating that certain practical problems of electromagnetic fields can be transformed, with the aid of conformal mapping, into equivalent ones convenient for actual computation. First, in the case of two-dimensional Laplacian potential problems of electrostatics, it is common to determine the scalar potential between two given conductor surfaces. If the given field is a polygonal domain, the mapping onto the half-plane is performed by a Schwarz-Christoffel transformation. But, in case a sharp corner exists, the field vector becomes infinite there. To avoid this inconvenience, a method of rounding a sharp corner may be available; the author thus proposes an empirical modification to be tried for this transformation. Next, in the case of Poisson potential problems, he selects the two-dimensional problems of electron flow in vacuum tubes under stationary conditions and problems of magnetic field within long conductors carrying parallel and uniformly distributed electric current flow, and then illustrates the possibility of simplifying, by means of conformal mapping, the boundary conditions prescribed on general cross-sectional configurations. Finally, in the case of propagation problems in wave guides, it is possible to replace an actual dynamic problem presented by the wave equation by an equivalent static one, if, in particular, the electromagnetic field can be expressed in terms of a scalar two-dimensional wave function and if the boundary conditions are of a certain linear homogeneous form. It then becomes easier to deal with this static problem when the given domain of the field is mapped conformally onto a canonical one. The method is illustrated here by an example. *Y. Komatu.*

Laudet, Michel. Potentiel et champ d'une lentille électrostatique cylindrique à trois fentes. Cahiers de Physique no. 41, 73–80 (1953).

On détermine d'abord analytiquement le potentiel et le champ dans le cas général d'une lentille cylindrique à électrodes planes et parallèles formée de six demi-plans opposés deux à deux. On étudie ensuite le cas particulier important de la lentille symétrique; on calcule numériquement le potentiel et le champ dans le plan de symétrie et on donne un abaque reliant les paramètres de la formule théorique aux constantes géométriques du système. Enfin, on indique la marche à suivre pour l'étude de l'objectif à immersion. *Author's summary.*

Bergmann, Otto. Conservation laws in classical electrodynamics. Physical Rev. (2) 90, 315 (1953).

***Carson, John R. Electric circuit theory and the operational calculus.** 2nd ed. Chelsea Publishing Company, New York, N. Y., 1953. x+197 pp. $3.95.

Reprinted by photo-offset from the first edition [McGraw-Hill, New York, 1926].

Synge, J. L. The fundamental theorem of electrical networks. Quart. Appl. Math. 11, 215 (1953).

Amplification of a proof in an earlier paper of the same title [same Quart. 9, 113–127 (1951); these Rev. 13, 189].

Gross, B. On linear electrical networks (preliminary note). Anais Acad. Brasil. Ci. 24, 443–447 (1952).

The problem treated is that of finding the indicial admittance of an electrical network whose steady state admittance function is given. The network is assumed to be constructed with an infinite number of lumped and distributed elements. The indicial admittance is first defined in terms of a contour integral, and upon invoking Carson's integral equation the given steady state admittance function is expressed in terms of a Cauchy integral in which the unknown function is related to the desired indicial admittance. The solution is obtained by inversion along the proper contour. The same method is applied to impedance functions. *R. Kahal* (Monterey, Calif.).

Weinberg, Louis. Synthesis of transfer functions with poles restricted to the negative real axis. J. Appl. Phys. 24, 207–216 (1953).

A rational fraction of the complex frequency variable s with poles restricted to the negative real axis and zeros located anywhere in the complex plane except on the positive real axis is realized to within a constant real factor as the transfer function of an unbalanced RC network. The synthesis procedure employs the Guillemin technique of

zero shifting and the paralleling of ladder structures but with an economy of ladders and elements by the use of a network partitioning theorem. *R. Kahal.*

Weinberg, Louis. Synthesis of unbalanced RLC networks. J. Appl. Phys. **24**, 300–306 (1953).

A method for the realization to within a constant factor of the transfer function of RLC networks with lossy coils, terminated with an arbitrary impedance, is given. The functions treated are those realizable with ladder structures (minimum phase) and must have numerator and denominator of degree not higher than third and fourth respectively. As is well known, the zeros and poles of minimum phase structures must lie in the left half complex frequency plane. No other restriction is placed on their location. The desired network is divided into two simpler networks by means of a partitioning theorem, one part being specified by a driving point admittance and a transfer admittance, the other by a driving point admittance and a transfer function. The synthesis of each part separately is presumably effected very simply. *R. Kahal* (Monterey, Calif.).

Gaponov, A. V. Nonholonomic systems of S. A. Čaplygin and the theory of commutator electrical machinery. Doklady Akad. Nauk SSSR (N.S.) **87**, 401–404 (1952). (Russian)

The paper considers electric networks made up of three-dimensional conductors [without capacitance?] in which the current distribution is known (making for a finite number of independent current parameters) and which have a finite number of mechanical freedoms. Without proofs and explanation of notations, the author claims that the equations of motion and current distribution assume the Lagrangian form, with Čaplygin-type corrective terms if commutators are present (in this case the electric constraints are non-holonomic). The specialization of these equations for an electric generator is indicated. *A. W. Wundheiler.*

Quantum Mechanics

Haag, Rudolf. Über die Objektivierbarkeit der Zustände in der nichtrelativistischen Quantenmechanik. Z. Naturforschung **8a**, 13–16 (1953).

de Broglie, Louis. Sur l'interprétation de la mécanique ondulatoire à l'aide d'ondes à région singulière. C. R. Acad. Sci. Paris **236**, 1453–1456 (1953).

Serebryanyĭ, R. V. The motion of the center of mass and the variation in width of wave packets. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **20**, 1130–1138 (1950). (Russian)

Explicit formulas, pertaining to rather well-known results, are given for the motion of the center of mass and the dispersion of free scalar and electromagnetic wave packets. *A. J. Coleman* (Toronto, Ont.).

Schönberg, M. A generalization of the quantum mechanics. Nuovo Cimento (9) **10**, 350–353 (1953).

To any wave function of a quantum mechanical system is associated by means of the inner product a linear functional over the Hilbert space of all wave functions. If the wave function is a solution of the time-dependent Schrödinger equation, the corresponding linear functional satisfies an equation which the author writes down. The author notes that the latter equation admits solutions which are non-linear functionals over the space of all wave functions and presents this remark as the basis of a generalization of quantum mechanics. The case of functionals expandable in Volterra series is considered and a rule of probabilistic interpretation of the formalism is proposed. A more detailed treatment is announced. *L. Van Hove.*

Barker, W. A., and Chraplyvy, Z. V. Conversion of an amplified Dirac equation to an approximately relativistic form. Physical Rev. (2) **89**, 446–451 (1953).

The Foldy-Wouthuysen transformation [Physical Rev. (2) **78**, 29–36 (1950)] is used to give relativistic form, to order $(\hbar/mc)(\partial/\partial x_\nu)$, to a linear equation of the Dirac type describing a fermion interacting with an arbitrary combination of scalar, vector, tensor, pseudovector and pseudoscalar fields. Tables are given containing all the pure and mixed interaction terms. As an example, quantum-electrodynamic terms are introduced phenomenologically into the fermion equation for the discussion of the Lamb-Retherford shift in the hydrogenic atom. *C. Strachan* (Aberdeen).

Cook, J. M. The mathematics of second quantization. Trans. Amer. Math. Soc. **74**, 222–245 (1953).

The author is concerned with second quantization for non-interacting particles, i.e., quantization of free linear fields. In part I, entitled "Mathematical preparation," the classes of operators used in second quantization are rigorously defined and studied in the framework of the theory of Hilbert space operators. This analysis is made in the Fock representation where the total number of particles is diagonal. In part II, entitled "Physical interpretation," simple cases of free fields (scalar mesons, photons) are used to show the physical meaning of the operators defined and properties established in part I. For the case of photons, the Hilbert space of quantum states of a single photon is constructed; the (physically meaningless) arbitrary gauge is, however, left in the description. One should also mention that the operators in Eq. (13) do not represent the infinitesimal rotations in ordinary space. A short account of the paper appeared previously [Proc. Nat. Acad. Sci. U. S. A. **37**, 417–420 (1951); these Rev. **13**, 410]. *L. Van Hove.*

Motz, Lloyd. Gauge invariance and classical electrodynamics. Physical Rev. (2) **89**, 60–66 (1953).

The author aims to give a divergence-free, classical, linear theory suitable to quantization. Some important ideas known in the literature are unified in this paper, e.g., the basic idea leading to the Lagrangian of Born-Infeld theory, the idea of gauge invariance in the sense of Weyl, later some concepts of Dirac's new electrodynamics. The decisive step constitutes the linearization of the theory by "rationalizing the generalized volume". The eqs. 45–46 thus obtained are linear and, as the author shows, lead to finite Coulomb energy. Some doubts can arise from the fact that the linearized Lagrange function is a column of four quantities. The action principle used by the author seems to require some further comments, as well as the question of consistency of eqs. 45–46 (which are $4+4\cdot 4=20$ in number with 8 variables). The derivation of finite Coulomb energy based on the discussion of approximate solutions seems to require more adequate proof. *J. Plebański* (Warsaw).

Sato, Iwao. On the formulation of the quantum field theory in the configuration space. Sci. Rep. Tôhoku Univ. Ser. I. **34**, 75–86 (1950).

The author works out in detail the configuration-space representation of the wave-functions describing an assembly of electrons, in the relativistic quantum theory of the Dirac electron-positron field. He follows the method which Fock [Z. Physik **75**, 622–647 (1932)] used for an assembly of non-relativistic particles, with only minor changes. By defining the wave-functions on a general space-like hypersurface, he is able to make all the equations manifestly relativistic. *F. J. Dyson* (Ithaca, N. Y.).

Klimontovič, Yu. L. Relativistic equation for the quantum distribution function. Doklady Akad. Nauk SSSR (N.S.) **87**, 927–930 (1952). (Russian)

The many-time formalism is used to obtain relativistically invariant equations equivalent to Liouville's theorem for the distribution function of a system of particles. *A. J. Coleman* (Toronto, Ont.).

Nishijima, Kazuhiko. On Lagrangian formalism. Progress Theoret. Physics **8**, 401–415 (1952).

Dyson's time-ordering symbol P is modified to apply to expressions involving first order derivatives. This enables the author to express the operator $S(t, t')$ directly in terms of the Lagrangian. The main body of the paper is devoted to a proof of the equivalence of this expression with that of Dyson in terms of the Hamiltonian. Using Peierl's procedure, this result is used to obtain commutation relations for field operators directly from the Lagrangian. *A. J. Coleman* (Toronto, Ont.).

Brown, G. E. Bound-state perturbation theory in four-dimensional momentum representation. Proc. Roy. Soc. London. Ser. A. **215**, 371–385 (1952).

Feynman methods and graphs are applied to the development of perturbation theory for bound states. Assuming all infinite terms removed, the author considers in turn the one-particle and two-particle problems. In the first, the external potential is treated exactly and radiative processes appear as perturbations. In the second, it is convenient to include in the zero order all successive interchanges of longitudinal photons between the two particles, with crossed exchanges appearing as a perturbation. The zero order then naturally includes also those pair production processes which may be deformed into ladder-type graphs. The effect of the exchange of two transverse photons in hydrogen is evaluated as 0.46 Mc/s for the 2S state, the effect of binding on the intermediate states being justifiably neglected. *H. C. Corben* (Pittsburgh, Pa.).

Scheidegger, A. E. Multiple quantization. Canadian J. Math. **5**, 26–36 (1953).

The author emphasizes that a careful investigation of the mathematical formulation of Quantum Theory shows that it can be regarded as a process which can be applied to almost any differential equation. Thus the quantization can be regarded as a corrective mathematical process which is applicable to unsatisfactory physical differential equations and the possibility of repeated or multiple quantization presents itself quite naturally. It can be guessed that each quantization makes the equations describing a physical process more and more accurate. In order to make a quantization procedure according to the canonical rules one needs a Hilbert vector $q(k, t)$ which depends on a time-like parameter. The Hilbert vector must be subject to a differential equation of Euler-Lagrangian type so that the canonical formalism can be set up. The next step is to proceed to the Heisenberg equations and hence to the Schrödinger equations of motion:

$$(*)\qquad \frac{\hbar}{i}\dot{\psi}(x)=H\psi(x),$$

"which complete the circle from a Hilbert vector to a Hilbert vector". There are several difficulties. First, the variable x is a continuous variable with an infinite range, whereas k in $q(k, t)$ assumes only a finite and discrete set of values. Furthermore, the ψ's are complex and the differential equation (*) contains only first order time-derivatives. Splitting (*) into its real and imaginary parts ($\psi=\psi_1+i\psi_2$) yields (if H is assumed to be real): $\hbar\dot{\psi}_1=-H\psi_2$, $\hbar\dot{\psi}_2=H\psi_1$, hence (*) becomes

$$(**)\qquad -\hbar^2\ddot{\psi}_1=HH\psi_1={}^2H\psi_1$$

and an identical equation for ψ_2. For example, the Schrödinger-Gordon equation is of the form (**). However the possibility exists of abandoning the correspondence between ψ and a canonical variable, which is indeed sometimes done in the quantum theory of fields. An interesting example demonstrating these ideas is given by the third quantization of the Schrödinger-Gordon equation. To achieve the third quantization, it is necessary to obtain a Schrödinger-representation for the Schrödinger-Gordon field, since the customary Heisenberg matrix-representation does not permit a further quantization. The energy eigenvalues of the third quantization of the Schrödinger-Gordon field are found to be $\frac{1}{2}\sum\hbar\omega_k N_k$. These eigenvalues are very similar to those which are obtained after the second quantization. The equations describe an ensemble of $\mathfrak{N}_k$ "particles" of the energy $\frac{1}{2}\hbar\omega_k$, k ranging over all the lattice points in momentum space. These "particles" have only half the energy of those of the Schrödinger-Gordon equation and the infinite term in the energy has disappeared (the lowest state is $\mathfrak{N}_k=0$ for every k). The author gives the interesting interpretation that his scheme can be taken as describing a Bose ensemble of fields (instead of an ensemble of particles) each of which satisfies the Schrödinger-Gordon equation. It is possible to proceed with the quantizations as often as we like, if we are able to find a recursion formula which connects the eigenvalues of ${}^nH$ in the relation

$$\dot{\phi}=-\frac{i}{\hbar}\,{}^nH\phi$$

with the eigenvalues of the Hamiltonian ${}^{n+1}H$ of the next quantization. The author obtains the result: the $(n+1)$th quantization describes a statistical ensemble of systems as described by the nth quantization, wherein the counting prescriptions of Einstein and Bose hold; the eigenvalues of the energy do not change any more after one has had a Schrödinger equation and quantized it once more. In the last section the author deals with anticanonical quantizations. The result is: the $(n+1)$th anticanonical quantization of a Schrödinger equation describes a statistical ensemble of such systems as are described by the nth Schrödinger equation. But now this statistical ensemble is a Fermi-Dirac ensemble. *M. Pinl* (Dacca).

Gupta, Suraj N. Quantum electrodynamics with auxiliary fields. Proc. Phys. Soc. Sect. A. **66**, 129–138 (1953).

By introducing auxiliary fields of particles of infinite mass, a method is developed for evaluating divergent inte-

grals in quantum electrodynamics. These particles are either described by the usual Lagrangian with its sign reversed or are supposed to obey the type of statistics opposite to that normally associated with their spin. It is argued that this is possible because these particles are supposed absent from the initial state and in virtue of their infinite mass could not be created in the final state; they therefore appear only in intermediate states and are accordingly unobservable. The second-order contribution to the self-energy of a free photon, computed by this somewhat artificial procedure, is then shown to vanish. *H. C. Corben* (Genoa).

Ivanenko, D., Kurdgelaidze, D., and Larin, S. Remarks on nonlinear mesodynamics. Doklady Akad. Nauk SSSR (N.S.) **88**, 245–247 (1953). (Russian)

In the equation $\Delta\varphi-k^2\varphi-\Lambda\varphi=-4\pi g\rho$, for the scalar meson function φ, the density ρ is replaced by its Thomas-Fermi approximation and the behaviour of the solution in the extreme relativistic and non-relativistic cases is briefly discussed. *A. J. Coleman* (Toronto, Ont.).

Kilmister, C. W. A new quaternion approach to meson theory. Proc. Roy. Irish Acad. Sect. A. **55**, 73–99 (1953).

In terms of the usual spinor terminology the author's results may be expressed as follows: A meson field is represented by a two-index four-component spinor ψ. Reasons are given for calling the meson wave equation $\gamma^s\partial\psi/\partial x^s=F\psi$, where γ^s are a set of Eddington E numbers (Dirac matrices) and F is a four index four component spinor. Special forms of F are taken to represent the field free case. The following well-known result is obtained: If ψ is restricted to be a symmetric spinor or an antisymmetric spinor, these equations reduce to the Kemmer form of the Proca equations for mesons of spin one and zero, respectively, for suitable choices of F, namely $Fx=\lambda x$ for arbitrary x.

A. H. Taub (Urbana, Ill.).

Friedrichs, K. O. Zur asymptotischen Beschreibung von Streuprozessen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 43–50 (1952).

The author previously investigated the analytical properties of the solution of the time-dependent Schrödinger equation for a Hamiltonian comprising a nonperturbed part with continuous energy spectrum and a perturbation satisfying convenient regularity conditions [Math. Ann. **115**, 249–272 (1938); Communications on Appl. Math. **1**, 361–406 (1948); these Rev. **10**, 547]. The results are here applied to a discussion of the time variation of the transition probability between stationary states of the unperturbed Hamiltonian. The transition probabilities between times $t=-\infty$ and $t=+\infty$ are shown to have definite asymptotic values, in agreement with the S-matrix theory. On the other hand, over a finite range of time, inversely proportional to the indetermination of the initial state in energy, the transition probabilities are found to be approximately proportional to the time, as predicted by quantum-mechanical perturbation theory. The two conventional descriptions are thus united and their respective domains of validity are exhibited.

L. Van Hove (Princeton, N. J.).

Baumann, Kurt. Eine einfache Herleitung der Streuformel von Bhabha. Acta Physica Austriaca **7**, 96–97 (1953).

Wick, G. C., Wightman, A. S., and Wigner, E. P. The intrinsic parity of elementary particles. Physical Rev. (2) **88**, 101–105 (1952).

Field quantities are usually assumed to obey an unambiguous transformation law under inversion of coordinates. This need not be true if the field quantity is not physically measurable. Here it is assumed that no physical measurement can distinguish between the state vectors $F_a+F_b+\cdots$, and $e^{i\alpha}F_a+e^{i\beta}F_b+\cdots$, where α, β are arbitrary phases and F_a, F_b, $\cdots$ are components of the state vector F along certain orthogonal subspaces in Hilbert space. This is not inconsistent with the superposition principle. These phase factors are unobservable if a "superselection" rule prevents spontaneous transitions between the subspaces and if no physically measurable quantities have non-zero matrix elements between the subspaces. Such a superselection rule is shown to hold between subspaces corresponding to integral and half-integral spin, to preserve invariance under time inversion. Gauge-invariance in field theory implies that parities of states with different charges cannot be compared. It will be possible to say that the parity of a particle of unit charge is a number of modulus unity. *C. Strachan.*

Caldirola, P., e Gulmanelli, P. Su una nuova equazione ondulatoria per una particella a spin 1/2. Nuovo Cimento (9) **9**, 834–845 (1952).

Corben [Nuovo Cimento (9) **9**, 580 (1952); these Rev. **14**, 228] has proposed a modified wave equation for particles of spin $\frac{1}{2}$ where the electric charge is represented by an operator. The authors apply this equation to the problem of a particle moving in a central field, and show that the eigenvalues obtained do not differ from those obtained with Dirac's equation, provided that an expression $(m_0^2+m_e^2)^{1/2}$ is interpreted as effective mass m of the particle. m_0, the mass of the "nude" particle, can be zero; m_e is the electromagnetic mass. Next, the equation is solved for a particle in a constant magnetic field. This leads to the conclusion that the particle has an additional magnetic moment $\lambda m_0/m$ (non-relativistic case) which will vanish with m_0. Finally, analogous results are obtained for a particle in a central field upon which a constant magnetic field is superposed. *E. Gora* (Providence, R. I.).

Gupta, K. K. On the Fierz-Pauli equation for particles of spin 3/2. Proc. Indian Acad. Sci., Sect. A. **35**, 255–264 (1952).

The Pauli-Fierz equation for particles of spin 3/2 are written in the form: $(P_k\alpha_k+\chi)\psi=0$. The commutation relations of the α_k are obtained and the expression in terms of the α_k of I^{kl} (the generators of the representation of the Lorentz group according to which ψ transforms) is given.

K. M. Case (Ann Arbor, Mich.).

Caldirola, Piero. È la massa dell'elettrone di natura elettromagnetica? Rend. Sem. Mat. Fis. Milano **22** (1951), 25–59 (1952).

In this paper a review is given of different attempts, developed by several authors during the last half-century, in order to give an electromagnetic interpretation of the electron mass. In particular, recent theories of Born and Infeld, of Dirac, and of Bopp are discussed.

Author's summary.

Moshinsky, Marcos. Transient effects in the dispersion caused by a rigid sphere. Revista Mexicana Fisica 1, 28–37 (1952). (Spanish. English summary)

A time-dependent treatment of the scattering by an impenetrable spherical potential is given. The transient phenomena are compared with those found by the author in the time-dependent treatment of single level nuclear scattering [Physical Rev. (2) **84**, 525–533 (1951); these Rev. **13**, 610]. The connection with the time-energy uncertainty relation is discussed. *K. M. Case* (Ann Arbor, Mich.).

Moshinsky, Marcos. Diffraction in time. Physical Rev. (2) **88**, 625–631 (1952).

To interpret the transient effects found in a previous time-dependent treatment of scattering [Moshinsky, Physical Rev. (2) **84**, 525–533 (1951); these Rev. **13**, 610], a problem in which a shutter in front of a beam of particles is suddenly opened is considered. Solutions are obtained assuming the "particles" obey the Schrödinger, wave, and Klein-Gordon equations. In the first case (and only here) the transient wave functions are found to resemble the functions that appear in Sommerfeld's theory of diffraction. *K. M. Case* (Ann Arbor, Mich.).

Moses, H. E. The canonical transformation for an electron-positron field coupled to a time-independent electromagnetic field. Physical Rev. (2) **89**, 115–122 (1953).

A canonical transformation is introduced to solve the problem of an electron-positron field coupled to an unquantized electromagnetic field constant in time. Using the Fock representation, a sequence of integro-difference equations are found which determine the transformation operator. It is shown how these equations may be solved in a perturbation theory manner. *K. M. Case.*

van Kampen, N. G. *S*-matrix and causality condition. I. Maxwell field. Physical Rev. (2) **89**, 1072–1079 (1953).

Well described in the author's summary: "The general aim is to obtain maximum information about the *S*-matrix with a minimum of assumptions concerning the interaction. This program is carried through for the scattering of the electromagnetic field by a fixed center. The center is assumed to be spherically symmetric and of finite size, so that the causality condition can be applied. From this condition it follows rigorously that the *S*-matrix has a one-valued analytic continuation, whose only singularities are poles in the lower half-plane, and whose behaviour at infinity can be specified. Particular consequences are (i) the analytic properties of Wigner's function R; (ii) the integral relation connecting real and imaginary parts of S; (iii) relations connecting the sum of the oscillator strengths with the scattering cross-section". The restriction to the electromagnetic field enables the author to make a more powerful use of the causality condition than was possible for previous writers who dealt with Schroedinger waves. *A. J. Coleman.*

Coester, F. The symmetry of the *S* matrix. Physical Rev. (2) **89**, 619–620 (1953).

It is proved that any Heisenberg *S*-matrix describing a collision between quantum-mechanical systems is symmetric for an appropriate choice of arbitrary phases in the wave functions in a representation where the square and one component of the total angular momentum are diagonal, and provided the Hamiltonian is invariant under time reversal (invariance of the *S*-matrix is implied by this). The consequences of this symmetry for the phases of the elements of S are discussed in the light of perturbation theory and the theory of the compound nucleus. *C. Strachan.*

Ivanenko, D., and Kolesnikov, N. The electrino hypothesis. Doklady Akad. Nauk SSSR (N.S.) **87**, 923–925 (1952). (Russian)

Encouraged by de Broglie's method of fusion and by evidence of the existence of a particle, the electrino, of half-life about 10^{-10} secs. and mass about twice that of the electron, the authors consider the possibility of explaining the β-decay spectrum as arising from the disintegration of electrinos. They find that the resulting theoretical energy profile agrees with observations for large energies of the emitted electron. *A. J. Coleman* (Toronto, Ont.).

Coester, F., and Jauch, J. M. Theory of angular correlations. Helvetica Phys. Acta **26**, 3–16 (1953).

Angular correlation functions between pure states have been calculated, but in experiment a statistical ensemble is observed corresponding to sums or averages over unobserved quantum numbers. A simplification of the mathematical treatment is thus achieved by the use of the density matrix ρ. An efficiency matrix ϵ is introduced and the most general angular correlation function is expressed as $W=\mathrm{Tr}(\epsilon\rho)$. An expression for W is obtained in terms of certain irreducible tensors. Successive radiations from a metastable nucleus, a nuclear reaction with two emergent particles, and internal conversion are discussed, and some rules are given for the construction of the irreducible tensors. *C. Strachan.*

Ekstein, H. Multiple scattering and radiation damping. II. Physical Rev. (2) **89**, 490–501 (1953).

In part I of this paper [Physical Rev. (2) **83**, 721–729 (1951); these Rev. **13**, 414] a momentum space matrix has been used to treat multiple elastic scattering by a system of fixed centers of force. This matrix is essentially Green's function of the single scattering problem, with the usual boundary conditions describing retarded waves. An alternative formulation with boundary conditions describing half-retarded, half-advanced (standing) waves is now proposed which makes it possible to use a more suitable approximation method. This is due to the circumstance that the influence of radiation damping upon the single scatterer is partly compensated by the reaction of the surrounding scatterers. Hence, it is advantageous to start with a description of the single scattering event from which radiation damping has been eliminated, and this is achieved with half-retarded, half-advanced waves. The theory of the propagation of matter waves in crystals and liquids is developed on this basis. *E. Gora* (Providence, R. I.).

Shimose, Tsuneto. On the retardation effects in the non-local field theory. Nat. Sci. Rep. Ochanomizu Univ. 1, 29–39 (1951).

In Bopp's theory of the electron [Z. Physik **125**, 615–628 (1949) and earlier papers] the Lagrange function and other quantities are represented by expansions in terms of the velocity of the electron and its higher derivatives. The higher order terms represent retardation effects and determine the internal structure of the electron. The author shows that such expansions can also be obtained from theories of the Tomonaga-Schwinger type if non-local commutation relations are used. Only the values of the expansion coefficients but not the structure of the terms depend

upon the particular choice of the "spreading out" functions in the commutation relations. The derivation of the corresponding generalizations of the Maxwell and Dirac equations will be given in another paper. *E. Gora.*

Kümmel, Hermann. Freie Elektronen in der unitären Quantenelektrodynamik. Z. Physik **134**, 78–94 (1952).

Summarizes the author's thesis which compares the unitary field theory of Ludwig [Z. Naturforschung **5a**, 637–641 (1950); these Rev. **12**, 783] with Feynman's theory of positrons. It is shown that the former theory easily gives the usual results for Møller scattering, bremsstrahlung, and the Compton effect. *A. J. Coleman* (Toronto, Ont.).

Watson, Kenneth M. Multiple scattering and the many-body problem—applications to photomeson production in complex nuclei. Physical Rev. (2) **89**, 575–587 (1953).

Multiple interaction of mesons within a complex nucleus is treated under the following assumptions: scattering arises from interaction of the meson with one nucleon at a time; absorption involves interaction with a pair of mesons (to conserve energy and momentum) at a time. The interactions are introduced in a phenomenological manner. The Lippmann-Schwinger formalism [Physical Rev. (2) **79**, 469–480 (1950); these Rev. **12**, 570] is used. The integral equation, which represents the Schroedinger equation there, is solved rigorously. The solution obtained reduces to expressions of the type suggested previously by different authors [see, for instance, M. Lax, Rev. Modern Physics **23**, 287–310 (1951); these Rev. **13**, 708] on grounds of physical plausibility, provided that nuclear excitation is neglected, and that the coordinates of the scatterers are considered as adiabatic parameters; the latter assumption amounts to using the impulse approximation. The expressions obtained permit decomposition into coherent and incoherent parts. It is thus made possible to investigate the validity of the "optical models" previously used, and to find systematic corrections to these models. The theory is applied to photomeson production. *E. Gora* (Providence, R. I.).

Ginsburg, V. L., and Tamm, I. E. On the theory of spin. Translated by G. Belkov. National Research Council of Canada, Ottawa, Tech. Translation **TT-305**, 23 pp. (1952).

Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **17**, 227–237 (1947); these Rev. **9**, 553.

el-Nadi, M. Sur la théorie du photon de L. de Broglie. J. Phys. Radium (8) **13**, 540–542 (1952).

The attempt of de Broglie to express the equations for a spin-one particle in terms of those for two particles of spin 1/2 is expressed here in another notation in which the wave function for unit spin appears as a 4×4 matrix.

H. C. Corben (Pittsburgh, Pa.).

Destouches-Aeschlimann, Florence. Intégrales opératorielles et extension de la mécanique ondulatoire. C. R. Acad. Sci. Paris **236**, 1140–1142 (1953).

Définition d'un système d'équations différentielles opératorielles; application à la mécanique ondulatoire; définition des intégrales, propriétés. Extension de la mécanique ondulatoire au cas de systèmes n'admettant pas d'hamiltonien. Exemples. Application à l'expérience de Millikan.

Author's summary.

Thermodynamics, Statistical Mechanics

Schmidt, Helmut. Eine einfache Herleitung der Verteilungsfunktionen für Bose- und Fermi-Statistik. Z. Physik **134**, 430–431 (1953).

The average occupation numbers of ideal Bose and Fermi gases are calculated by comparing systems with $N-1$ and N particles (N large). *L. Van Hove.*

Klein, G., et Prigogine, I. Sur la mécanique statistique des phénomènes irréversibles. I. Physica **19**, 74–88 (1953).

The authors recall the known equilibrium properties of a one-dimensional system of classical particles with interaction between nearest neighbors, in particular, the validity of the so-called superposition principle, asserting that the distribution function for three neighboring particles is a product of pair distribution functions. Following Born and Green [Proc. Roy. Soc. London. Ser. A. **190**, 455–474 (1947); these Rev. **9**, 402] and in a way similar to Bogolyubov [Acad. Sci. USSR. J. Phys. **10**, 265–274 (1946); these Rev. **9**, 72] and Gurov [Dissertation, Moscow, 1946], they assume the superposition principle to retain its validity for a large class of non-equilibrium distributions. The results of part II of the present investigation, reviewed below, will show that this assumption does not give a satisfactory theory of irreversible processes.

From the superposition principle the authors derive the partial differential equation describing the time variation of the pair distribution function for conditions slightly different from equilibrium. This equation is seen to have stationary (time-independent) solutions different from the equilibrium solution. *L. Van Hove* (Princeton, N. J.).

Klein, G., et Prigogine, I. Sur la mécanique statistique des phénomènes irréversibles. II. Physica **19**, 89–100 (1953).

The equations derived in the preceding paper are applied to the discussion of three non-equilibrium processes: starting at time $t=0$ from initial conditions where the distances between particles have their equilibrium distribution but the velocity distribution is perturbed, the pair distribution function is expanded in powers of t. The term in t vanishes and the term in t^2 indicates an approach toward the equilibrium distribution. The same conclusion holds for an initial distribution in which the total energy is unequally shared between the kinetic and potential terms. In both cases, however, the vanishing of the term in t indicates that the initial conditions are of particular form; the authors also mention without proof that the entropy always remains constant instead of increasing toward its maximum (equilibrium) value.

The case of thermal conductivity is then studied, with the conclusion that the equation for the pair distribution function has no stationary solution in presence of a temperature gradient. An argument is presented showing that the same conclusion holds for a three-dimensional system if the superposition principle is assumed outside of equilibrium. The inadequacy of this principle to describe irreversible processes is thus demonstrated. *L. Van Hove* (Princeton, N. J.).

McLellan, A. G. The stress tensor, surface tension and viscosity. Proc. Roy. Soc. London. Ser. A. **217**, 92–96 (1953).

The contribution of the intermolecular forces to the stress tensor of a fluid is calculated in terms of the pair-distribution

function. Both the result and the method are identical to those given earlier by J. H. Irving and J. G. Kirkwood [J. Chem. Phys. 18, 817–829 (1950), Appendix; these Rev. 12, 230]. The result is applied to the calculation of the viscosity of the fluid and the tension of a spherical surface.

L. Van Hove (Princeton, N. J.).

Hoffmann, T. A. Some investigations in the field of the theory of solids. III. Plane and space lattice of similar atoms. Acta Phys. Acad. Sci. Hungar. 2, 97–106 (1952). (Russian summary)

Hoffmann, T. A. Some investigations in the field of the theory of solids. IV. A–B-type ordered binary systems in the plane and the space. Acta Phys. Acad. Sci. Hungar. 2, 107–127 (1952). (Russian summary)

In two previous papers [same Acta 1, 5–35, 175–195 (1951); these Rev. 13, 308, 714] Hoffmann has built up the theory for the properties of the chain structure of similar and dissimilar atoms as a problem of the linear combination of atomic orbitals. In the two papers under review this material is used to calculate the energy and density of states in a square plane array and in a simple cubic lattice made up of atoms of one kind (III) and of atoms of two kinds (IV) in a binary system. A comparison of the energy distributions of the density of states is made for the linear, plane, and spatial cases. *R. Truell* (Providence, R. I.).

Hoffmann, T. A. Some investigations in the field of the theory of solids. V. Adsorption. Surface states. Acta Phys. Acad. Sci. Hungar. 2, 195–208 (1952). (Russian summary)

Using the background developed in the preceding four papers of this series, the author now deals with the effect of a surface layer of foreign atoms on the surface of a crystal of atoms of one type. The effects of a single atom layer are considered in connection with adsorption, surface states, etc. The author obtains energy levels outside the main energy band for the bulk material. The discussion of surface states is carried out in terms of the number and position of these levels above and below the main band. It is pointed out that surface levels do not always occur in pairs if the calculations are properly carried out. *R. Truell.*

BIBLIOGRAPHICAL NOTES

Atti del Quarto Congresso dell'Unione Matematica Italiana.

These Atti are published in two volumes in the Edizione Cremonese of the Casa Editrice Perrella [Rome, 1953]. The congress was held in Taormina, 25–31 October, 1951. The first volume (324 pp.) contains the general lectures, the second volume (684 pp.) contains short reports. The papers in the second volume which are being or have already been published in full elsewhere will not be reviewed separately.

Comptes Rendus du Premier Congrès des Mathématiciens Hongrois.

These Comptes Rendus are published by the Akademiai Kiadó [Budapest, 1952] in a volume of 789 pages. The congress was held 27 August–2 September, 1950 under the auspices of the Société Mathématique János Bolyai. Individual papers will be reviewed separately.

Bulletin de l'Académie Polonaise des Sciences, Classe Troisième.

The Classe Troisième of this Bulletin will contain papers in mathematics, astronomy, physics, chemistry, geology and geography. Vol. 1, nos. 1–2, is dated 1953. This publication will replace the Bulletin International de l'Académie Polonaise des Sciences et des Lettres, Classe des Sciences Mathématiques et Naturelles, which terminated with the March-June issue for 1951.

Commentarii Mathematici Universitatis Sancti Pauli.

Vol. 1, no. 1, of this journal is dated December, 1952. It carries also a title in Japanese, Rikkyô Daigaku Sûgaku Zassi, and is published by the Rikkyô Daigaku, Ikebukuro, Tokyo.

Czechoslovak Mathematical Journal.

Publication of this journal was suspended with vol. 1, no. 4. Starting with vol. 2, no. 1, the Russian version, Čehoslovackiĭ Matematičeskiĭ Žurnal, will contain summaries in English, French, or German.

Journal of the Gakugei College, Tokushima University.

Volume 1, with the subtitle Mathematics, is dated 1950; vol. 2, with subtitle Natural Science, is dated February 1952. The main title also varies. It is published by Tokushima University, Tokushima, Japan.

Journal of Mathematics.

Vol. 1, no. 1, of this journal is dated September, 1951. It is published by the Metropolitan Institute for Mathematics, Tokyo Metropolitan University, Meguro, Tokyo.

Kumamoto Journal of Science. Series A (Mathematics Physics and Chemistry).

Vol. 1, no. 1, is dated October 1952. It is published by the Faculty of Science, Kumamoto University, Kumamoto, Japan.

Mathematica Scandinavica.

Vol. 1, no. 1, appeared in August 1953. It is published in Copenhagen jointly by the Mathematical Societies of Denmark, Finland, Iceland, Norway and Sweden. Correspondence should be addressed to Mathematica Scandinavica, Blegdamsvej 15, København Ø, Denmark. This journal, together with Nordisk Matematisk Tidskrift, replaces the Norsk Matematisk Tidsskrift (which terminated with vol. 34, no. 4) and the Matematisk Tidsskrift, Series A and B (both of which terminated in 1952 with heft 4 and 3–4, respectively).

Nordisk Matematisk Tidskrift.

Vol. 1, no. 1, appeared in 1953. It is published in Oslo jointly by a number of Scandinavian organizations devoted to instruction and research in mathematics. Correspondence should be addressed to Matematisk Institutt, Blindern, Oslo, Norway.

Rozprawy Matematyczne.

The first issue of this journal is dated 1952. Each issue is devoted to one paper. It is published by the Polskie Towarzystwo Matematycznę, Śniadeckich 8, Warsaw.

The Science Reports of the Saitama University. Series A (Mathematics, Physics and Chemistry).

Vol. 1, no. 1, is dated 1952. It is published by the Saitama University, Urawa, Japan.

Zastosowania Matematyki.

Vol. 1, no. 1, is dated 1953. As the title indicates, this journal will be devoted to applications of mathematics. All papers of the first number are in Polish. It is published by the Polskie Towarzystwo Matematyczne, Śniadeckich 8, Warsaw.

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From review by H. Behnke, Bull. Amer. Math. Soc., January 1952.

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